MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 8
Due: April 15, 2005

Homework Assignment #8 (Final Version)

1. Read Edwards, Sections 2.4–2.8.
   (See also Schneider and Barker, Sections 7.5–7.6.)

2. (*) Find the points on the line $x + y = 10$ and the ellipse $x^2 + 2y^2 = 1$ which are closest.

3. (*) Use critical point classification and the method of Lagrange multipliers to find the point(s) on the closed unit sphere

   $$D^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$$

   (which includes its interior) where the function

   $$f(x, y, z) = x^3 + y^3 + z^3$$

   attains its maximum and minimum.

4. (A) Given positive real numbers $x_1, \ldots, x_n$, we define their arithmetic and geometric means as follows:

   $$A.M. = \frac{x_1 + \cdots + x_n}{n}$$

   $$G.M. = \sqrt[n]{x_1 \cdots x_n}$$

   Use Lagrange multipliers to prove that the geometric mean is always less than or equal to the arithmetic mean by minimizing the function $f(x_1, \ldots, x_n) = \frac{1}{n}(x_1 + \cdots + x_n)$ on the set $S = \{\mathbf{x} \in \mathbb{R}^n \mid g(\mathbf{x}) = 0\}$, where $g(x_1, \ldots, x_n) = x_1 \cdots x_n - 1$.

5. (B) Taken from Edwards p. 158, problem #8.2 and 8.3:

   Let $q(x, y, z) = 2x^2 + 5y^2 + 2z^2 + 2xz$ be a quadratic form. Show that q is positive-definite by:

   (a) using Theorem 8.8.

   (b) diagonalizing the quadratic form.

6. (C) Let $f(x, y, z) = xy^2z^3$, and consider the point $\mathbf{a} = (1, 0, -1)$. Find the second-order Taylor polynomial T_2 for f at $\mathbf{a}$, and show directly that the second-order remainder, defined as $R_2(\mathbf{h}) = f(\mathbf{a} + \mathbf{h}) - T_2(\mathbf{h})$, satisfies:

$$\lim_{||\mathbf{h}|| \to 0} \frac{R_2(\mathbf{h})}{||\mathbf{h}||^2} = 0$$

Alternatively, if we think of T_2 as a function of (x, y, z), we can define $R_2(x, y, z) = f(x, y, z) - T_2(x, y, z)$ and show that:

$$\lim_{(x,y,z) \to (1,0,-1)} \left(\frac{R_2(x, y, z)}{(x-1)^2 + y^2 + (z+1)^2} \right) = 0$$

7. (*) Let $f(x, y) = x^2 \sin y$, and consider the point $\mathbf{a} = (3, \frac{\pi}{2})$. Find the n-th order Taylor polynomials T_n for f at $\mathbf{a}$ when $n = 0, 1, 2, 3$, and express the $T_n(x, y)$ as polynomials in $(x - 3)$ and $(y - \frac{\pi}{2})$. How does T_3 compare with the third-order Taylor polynomials for $g(x) = x^2$ at $a = 3$ and for $h(y) = \sin y$ at $b = \frac{\pi}{2}$? Can you predict T_4 for f at $\mathbf{a}$ in terms of the fourth-order Taylor polynomials for g and h (at a and b, respectively)?

8. (D) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be of class C^3. Let $\mathbf{a}$ be a critical point of f, that is, $\nabla f(\mathbf{a}) = \mathbf{0}$. Show that if the quadratic form $q(\mathbf{h})$ corresponding to f at $\mathbf{a}$ is positive-definite, then $f(\mathbf{a})$ is a local minimum.

   (We are looking for a formal proof using appropriate neighborhoods defined by deltas and epsilons using the equation:

   $$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) = q(\mathbf{h}) + R_2(\mathbf{h}).$$

   You may use Taylor's Theorem without proof.)