MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 9
Due: April 22, 2005

Homework Assignment #9 (Final Version)

1. Read Edwards, Sections 4.1–4.3.

2. (A) Construct a set of charts and verify the conditions to show that the set $S^3 = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbb{R}^4$ is a 3-manifold.

3. (B) In class, we proved the following theorem:

   Theorem. Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \longrightarrow \mathbb{R}$ be a bounded function. If f is continuous at a, then $o(f, a) = 0$.

   Prove the converse. (Recall that $o(f, a)$ is the *oscillation* of f at a.)

4. (B) Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \longrightarrow \mathbb{R}$ be a bounded function. If P is a partition of A and P' is a refinement of P, show that
$$L(f, P) \leq L(f, P').$$

5. (*) Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \longrightarrow \mathbb{R}$ be a bounded function. If P_1 and P_2 are any two partitions of A, show that
$$L(f, P_1) \leq U(f, P_2).$$

6. (C) Let $f : A \longrightarrow \mathbb{R}$ be integrable. Show that $|f|$ is integrable and that
$$\left| \int_A f \right| \leq \int_A |f|.$$

7. (*) Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let
$$I = \{f : A \longrightarrow \mathbb{R} \mid f \text{ is integrable on } A\}.$$

   (a) Show that I is a vector space over $\mathbb{R}$ by showing that if $f_1, f_2 \in I$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2 \in I$. (In other words, I is a subspace of the vector space $V = \{f : A \longrightarrow \mathbb{R}\}$.)

   (b) Show that $\int_A cf = c \int_A f$ and $\int_A (f_1 + f_2) = \int_A f_1 + \int_A f_2$.

8. (D) Let $A = [0,1] \times [0,1]$, and define $f : A \longrightarrow \mathbb{R}$ as follows:

$$f(x,y) = \begin{cases} 0 & , \text{ if } x \notin \mathbb{Q} \\ 0 & , \text{ if } x \in \mathbb{Q} \text{ and } y \notin \mathbb{Q} \\ \frac{1}{q} & , \text{ if } x \in \mathbb{Q} \text{ and } y = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

(a) For each $\mathbf{a} \in A$, determine $o(f, \mathbf{a})$.

(b) Show that f is integrable on A. (What is $\int_A f$?)

(For comparison, look back at problem #1.5.)

9. (*) Give an example of a closed set of measure zero that does not have content zero.

10. (deferred)

Let $A = \{x \in [0,1] \mid \text{the decimal expansion of } x \text{ has no 8's}\}$.

Let $B = \{n \in \mathbb{N} \mid \text{the decimal expansion of } n \text{ has no 8's}\}$.

(a) Find the content of A.

(b) Decide whether the infinite series $\displaystyle\sum_{n \in B} \frac{1}{n}$ converges or diverges.

11. (deferred)

For a function $f : [0,1] \longrightarrow \mathbb{R}$, let $A = \{x \in [0,1] \mid f \text{ is not differentiable at } x\}$. Find such an f satisfying the following conditions:

- f is continuous.
- $f(0) = 0$
- $f(1) = 1$
- A has content zero.
- If $x \notin A$, then $f'(x) = 0$.

(Hint: Use the Cantor set.)