MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm Solutions (in-class portion)

1. **Properties of Number Systems**

   For each entry in the following table, identify whether the number system at the head of the column has that property or not by placing a "Y" for yes and an "N" for no.

Property	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{Z}/6\mathbb{Z}$
Existence of Additive Inverses	N	Y	Y	Y	Y
Existence of Multiplicative Inverses	N	N	Y	Y	N
Ordered	Y	Y	Y	Y	N
Well-ordered	Y	N	N	N	N
Complete	X	Y	N	Y	X

2. **True or False**

   (a) **T** or **F** **If $L : V \longrightarrow V$ is an injective linear map, then it is surjective.**

   False. We have seen examples (such as the shift operator on the space of infinite sequences) that are injective without being surjective. (The statement *is* true if $dim(V) < \infty$.)

   (b) **T** or **F** **If $L : V \longrightarrow W$ is a bijective linear map, then $V \cong W$.**

   True. This is the definition of isomorphic.

   (c) **T** or **F** **If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a set of linearly independent vectors, then any $\mathbf{v} \in V$ may be written as a linear combination of these vectors in a unique way.**

   False. If a vector could be written as such a linear combination, the representation would be unique, but the set of vectors might not span V.

(d) **T** or **F** **If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a set of vectors such that none of them is a scalar multiple of any of the others, then the set is linearly independent.**

False. It is possible to have a non-trivial linear combination giving zero even if no two are scalar multiples of each other.

(e) **T** or **F** **If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a set of vectors such that span$\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = V$, then $dim(V) \leq n$.**

True. This is a theorem from class.

(f) **T** or **F** **If $L : V \longrightarrow V$ is a linear transformation, then $Ker(L)$ is a subspace of V.**

True. This is a theorem from class.

(g) **T** or **F** **The set of vectors $\{(0,1,2),(1,1,1),(0,0,0)\}$ is linearly independent in $\mathbb{R}^3$.**

False. No set of vectors including $\mathbf{0}$ can be linearly independent. In this example, we have $0 \cdot (0,1,2) + 0 \cdot (1,1,1) + 1 \cdot (0,0,0) = (0,0,0)$, even though not all the coefficients are 0.

(h) **T** or **F** **If $V =$ span$\{(0,1,2,3,4),(1,1,1,1,1)\}$, then $dim(V) = 5$.**

False. Since the two vectors are not scalar multiples of each other, they are linearly independent, and $dim(V) = 2$.

3. Let $\{a_n\}$ and $\{b_n\}$ be sequences of rational numbers.

(a) Define what it means for $\{a_n\}$ to be a Cauchy sequence.
The sequence $\{a_n\}$ is Cauchy means that, given any $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ whenever $n, m > N$.

(b) If $\{a_n\}$ and $\{b_n\}$ are both Cauchy sequences, show that $\{a_n \cdot b_n\}$ (the term-by-term product of the original two sequences) is also a Cauchy sequence.

You may either cite the following lemma from class or prove it directly as follows:
Lemma Any Cauchy sequence $\{a_n\}$ is bounded. That is, there exists some $M \in \mathbb{R}$ such that $|a_n| < M$, for every $n \in \mathbb{N}$.
Proof: Since the sequence is Cauchy, there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ whenever $n, m > N$. This implies that for every $n > N$,

$$|a_n - a_{N+1}| < 1$$

and hence

$$-1 < a_n - a_{N+1} < 1$$

or

$$a_{N+1} - 1 < a_n < a_{N+1} + 1$$

which implies that $|a_n| < \max\{|a_{N+1} - 1|, |a_{N+1} + 1|\}$ for every $n > N$. Finally, let

$$M = \max\{|a_1|, |a_2|, \dots, |a_N|, |a_{N+1} - 1|, |a_{N+1} + 1|\}.$$

This M bounds the sequence.

Now, we move on to the proof of the statement. Let M_1 and M_2 be the bounds for the sequences $\{a_n\}$ and $\{b_n\}$, respectively, as given in the lemma.
Given $\varepsilon > 0$, choose $N = \max\{N_1, N_2\}$, where N_1 is such that $|a_n - a_m| < \frac{\varepsilon}{2M_2}$, for all $n, m > N_1$, and N_2 is such that $|a_n - a_m| < \frac{\varepsilon}{2M_1}$, for all $n, m > N_2$. Then:

$$\begin{aligned} |a_n \cdot b_n - a_m \cdot b_m| &= |(a_n \cdot b_n - a_m \cdot b_n) + (a_m \cdot b_n - a_m \cdot b_m)| \\ &= |b_n||a_n - a_m| + |a_m||b_n - b_m| \\ &= M_2 \cdot \tfrac{\varepsilon}{2M_2} + M_1 \cdot \tfrac{\varepsilon}{2M_1} \\ &= \varepsilon, \end{aligned}$$

where the inequality follows from the fact that $n, m > N$.

4. Let $T : V \to V$ be a non-zero linear map. We consider the situation when $T^2 = 0$.

(Note that $T^2 = T \circ T$, in other words, T composed with itself, and 0 is the zero linear map that takes every vector in V to $\mathbf{0}$.)

(a) Show that $T^2 = 0$ if and only if $Im(T) \subset Ker(T)$.

($\Rightarrow$) Assume $T^2 = 0$.

Suppose $\mathbf{v} \in Im(T)$. Then there is some $\mathbf{w} \in V$ such that $T(\mathbf{w}) = \mathbf{v}$. By the assumption, we see that $T(\mathbf{v}) = T(T(\mathbf{w})) = T^2(\mathbf{w}) = 0$, and hence $\mathbf{v} \in Ker(T)$.

($\Leftarrow$) Assume $Im(T) \subset Ker(T)$.

Consider any $\mathbf{v} \in V$. We need to show that $T^2(\mathbf{v}) = 0$. Note that $T(\mathbf{v}) \in Im(T)$ and that therefore $T(\mathbf{v}) \in Ker(T)$, by assumption. This implies that $T(T(\mathbf{v})) = 0$, which is what we needed.

(b) Show that if $T^2 = 0$ and $dim(V) = 3$, then $dim(Ker(T)) = 2$.

By the Rank-Nullity Theorem, we know that $dim(Ker(T)) + dim(Im(T)) = dim(V) = 3$, and by the result from part (a), since $Im(T) \subset Ker(T)$, it follows that $dim(Im(T)) \leq dim(Ker(T))$. Putting these together, we see that $2 \cdot dim(Im(T)) \leq 3$, which means that $dim(Im(T)) = 0$ or 1, but since T is supposed to be non-trivial, in fact, $dim(Im(T)) = 1$, and hence $dim(Ker(T)) = 2$.

(c) Give an example of a non-trivial vector space V and a non-trivial linear map $T : V \to V$ satisfying $T^2 = 0$.

Three examples:

1. $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (0, x)$
2. $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x - y, x - y)$
3. $D : P_1 \to P_1$, which is the usual differentiation operator such that $D(ax + b) = a$.

5. Let $P_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 | a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ be the vector space of polynomials of degree three or less with real coefficients. Consider the linear map $I : P_3 \longrightarrow \mathbb{R}$ defined by $I(p(x)) = \int_0^1 p(x)\, dx$, for $p(x) \in P_3$.

(a) Find $dim(Ker(I))$.

By the Rank-Nullity Theorem, we know that

$$dim(Im(I)) + dim(Ker(I)) = dim(P_3) = 4.$$

Since $dim(Im(I)) \subset \mathbb{R}$, we see that $dim(Im(I)) \leq 1$. Since $I(2cx) = \int_0^1 2cx\, dx = cx^2\big|_0^1 = c$, we see that any $c \in \mathbb{R}$ is also in $Im(I)$, and hence $Im(I) = \mathbb{R}$, or $dim(Im(I)) = 1$.

Thus, $dim(Ker(I)) = 3$.

(b) Exhibit a basis for $Ker(I)$.

Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in Ker(I)$. Then

$$\begin{aligned} I(p(x)) &= \int_0^1 (a_0 + a_1x + a_2x^2 + a_3x^3)\, dx \\ &= a_0x + \tfrac{1}{2}a_1x^2 + \tfrac{1}{3}a_2x^3 + \tfrac{1}{4}a_3x^4\big|_0^1 \\ &= a_0 + \tfrac{1}{2}a_1 + \tfrac{1}{3}a_2 + \tfrac{1}{4}a_3 \\ &= 0 \end{aligned}$$

Choosing $a_0 = 1$ and $a_1 = -2$ yields one non-zero polynomial $p_1(x) = 1 - 2x$.

Choosing $a_0 = 1$ and $a_2 = -3$ yields another $p_1(x) = 1 - 3x^2$.

Finally, choosing $a_0 = 1$ and $a_3 = -4$ yields $p_1(x) = 1 - 4x^3$.

(c) Prove that your answer to part (b) is in fact a basis.

We check (using the degrees of these polynomials) that these three vectors are linearly independent. Suppose $c_1 \cdot p_1(x) + c_2 \cdot p_2(x) + c_3 \cdot p_3(x) = 0$. Then $(c_1 + c_2 + c_3) + (-2c_1)x + (-3c_2)x^2 + (-4c_3)x^3$ is the zero polynomial, but the only way this can be true is if $c_1 = c_2 = c_3 = 0$.

Since any three linearly independent vectors form a basis, we are done.

6. Let V be a vector space over the field F, and let W be a subspace of V. We define the quotient space V/W as follows:

$$V/W = \{\mathbf{v} + W | \mathbf{v} \in V\}/\sim,$$

where $\mathbf{v}_1 + W \sim \mathbf{v}_2 + W$ if and only if $\mathbf{v}_1 - \mathbf{v}_2 \in W$.

The elements of the quotient space are called cosets, and they have the form

$$\mathbf{v} + W = \{\mathbf{v} + \mathbf{w} | \mathbf{w} \in W\}.$$

If we define addition and scalar multiplication as follows:

$$(\mathbf{v}_1 + W) + (\mathbf{v}_2 + W) = (\mathbf{v}_1 + \mathbf{v}_2) + W$$

$$c \cdot (\mathbf{v} + W) = (c \cdot \mathbf{v}) + W$$

for any $c \in F$ and any $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V$, then in fact, the quotient space V/W is a vector space over F.

(a) **It is a fact that $\sim$ is an equivalence relation. Show that $\sim$ is *transitive*.**

Suppose $\mathbf{v}_1 + W \sim \mathbf{v}_2 + W$ and $\mathbf{v}_2 + W \sim \mathbf{v}_3 + W$. Then by the definition of equivalence, these imply that $\mathbf{v}_1 - \mathbf{v}_2 \in W$ and $\mathbf{v}_2 - \mathbf{v}_3 \in W$, respectively. Since W is a subspace, it is closed under addition, and hence $\mathbf{v}_1 - \mathbf{v}_3 = (\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{v}_2 - \mathbf{v}_3) \in W$. Again by the definition of equivalence, this implies that $\mathbf{v}_1 + W \sim \mathbf{v}_3 + W$.

(b) **It is a fact that addition and scalar multiplication are well-defined. Show that *addition* is well-defined.**

Suppose $\mathbf{v}_1 + W \sim \mathbf{v}_2 + W$ and $\mathbf{u}_1 + W \sim \mathbf{u}_2 + W$. Then by the definition of equivalence, these imply that $\mathbf{v}_1 - \mathbf{v}_2 \in W$ and $\mathbf{u}_1 - \mathbf{u}_2 \in W$, respectively. Again, since W is a subspace, it is closed under addition, and hence $(\mathbf{v}_1 + \mathbf{u}_1) - (\mathbf{v}_2 + \mathbf{u}_2) = (\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{u}_1 - \mathbf{u}_2) \in W$. By the definition of equivalence, this implies that $(\mathbf{v}_1 + \mathbf{u}_1) + W \sim (\mathbf{v}_2 + \mathbf{u}_2) + W$, and hence addition is well-defined.

(c) **It is a fact that V/W is a vector space over F. Show that V/W satisfies axiom V3 concerning *additive identities*.**

It is easy to check that $\mathbf{0} + W$ is the additive identity since $(\mathbf{0} + W) + (\mathbf{v} + W) = (\mathbf{0} + \mathbf{v}) + W = \mathbf{v} + W$, and $(\mathbf{v} + W) + (\mathbf{0} + W) = (\mathbf{v} + \mathbf{0}) + W = \mathbf{v} + W$ as well.