Last Name: ______________________________

First Name: ______________________________

MATH 23a, FALL 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm (in-class portion)
October 29, 2004

Directions: You have fifty minutes for this exam. No calculators, notes, books, etc. are allowed. Please answer on the pages provided. Show work in order to allow for partial credit.

Problem	Points	per part	Score
1	38	2 each	
2	12		
3	16	4 each	
4	16	4/4/8	
5	15	5 each	
Total	97	97	

1. **True or False**

T or **F** The successor of any natural number is another natural number.

T or **F** The integers form an ordered field.

T or **F** The rational numbers are well-ordered.

T or **F** $\mathbb{Z}/n\mathbb{Z}$ is a commutative ring for any integer $n \geq 2$.

T or **F** In any ordered ring with identity, $0 < 1$.

T or **F** According to the equivalence relation that we used in class to define the integers, the two pairs of natural numbers, $(1, 3)$ and $(2, 6)$, are equivalent.

T or **F** A field is a vector space over itself.

T or **F** The set $\{\mathbf{0}\}$ is a vector space over any field.

T or **F** The vector space $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

T or **F** If $V = (\mathbb{Z}/p\mathbb{Z})^n$ with p prime and $n \geq 2$, then V has $(p^n - 1)/(p - 1)$ distinct 1-dimensional subspaces.

T or **F** If $V = span\{\mathbf{v}_1, \mathbf{v}_2\}$, then $dim(V) = 2$.

T or **F** If $dim(V) = n$, then any set of $n+1$ (or more) vectors in V must span V.

T or **F** The set of functions $\{1, \cos x, \cos 2x\}$ is linearly independent, considered as a subset of $C[0, 2\pi]$.

T or **F** If a set of vectors spans V, then that set contains a basis for V.

T or **F** If $L : V \to W$ is linear and injective, then $dim(Im(L)) = dim(V)$.

T or **F** If $L : V \to W$ is linear and surjective, then L is invertible.

T or **F** If $L : V \to W$ is linear and bijective, then $V \cong W$.

T or **F** If $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a linearly dependent set in the vector space U, and $W = span\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, then $dim(W) < n$.

T or **F** $P_3(\mathbb{R}) \cong \mathbb{R}^4$, where $P_3(\mathbb{R})$ is the collection of polynomials with real coefficients and degree less than or equal to 3.

2. Let V be a vector space over the field F, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. Let $\mathbf{w} \in V$ be a randomly chosen vector. Prove that there exists a unique way to write $\mathbf{w}$ as a linear combination of the vectors in the basis.

3. Consider the field $F = \mathbb{Z}/7\mathbb{Z}$ and the vector space $V = (\mathbb{Z}/7\mathbb{Z})^2$.
Let $L : V \to V$ be the linear map given by:

$$L(x, y) = (2x + 5y, x + 6y)$$

(For simplicity throughout the problem, you may denote the elements of F by 0,1,2,3,4,5, and 6 rather than with the usual equivalence class notation.)

(a) Write down *all* of the subspaces of V.

(b) Identify $Ker(L)$ as one of the subspaces of V from part (a).

(c) Identify $Im(L)$ as one of the subspaces of V from part (a).

(d) What do the results of parts (b) and/or (c) say about the invertibility of L?

4. On the homework (problem #3.4), we considered the vector space of real-valued functions whose domain was a finite set S and showed that the dimension of this vector space was equal to the cardinality of S. That is, if $S = \{a_1, a_2, \ldots, a_n\}$ and $V = \{f : S \to \mathbb{R}\}$, then $dim(V) = n$.

   For this problem, consider the special case $S = \{a, b, c, d, e\}$, and once again let $V = \{f : S \to \mathbb{R}\}$. Now, let

$$W = \{f \in V \mid f(a) + f(b) = 0 \text{ and } f(d) - f(e) = 0\}.$$

   (a) Show that W is a subspace of V.

   (b) Write down a basis for W.

   (c) Show that your answer to part (b) is, in fact, a basis for W. *(An incorrect answer, especially an over-simplified one, for part (b) may prevent full points on part (c).)*

5. In this problem, we consider the First Isomorphism Theorem and its proof.

Theorem. (The First Isomorphism Theorem)
Let $L : V \to W$ be linear and surjective. Then:

$$V/Ker(L) \cong W.$$

Of course, we need to explain our notation. Here, $V/Ker(L)$ is the set of equivalence classes of vectors in V, with the equivalence relation $\sim$ defined as follows:

$$\mathbf{u} \sim \mathbf{v} \text{ provided that } \mathbf{u} - \mathbf{v} \in Ker(L)$$

In other words, an equivalence class $[\mathbf{u}] \in V/Ker(L)$ is a set of the form:

$$[\mathbf{u}] = \{\mathbf{v} \in V \mid \mathbf{u} - \mathbf{v} \in Ker(L)\}$$

We will show in class next week that, in fact, $V/Ker(L)$ is a vector space, but for this problem, you may assume that it is.

Proof:
Define $\varphi : V/Ker(L) \to W$ as follows:

$$\varphi([\mathbf{u}]) = L(\mathbf{u})$$

(a) Show that φ is well-defined.
(b) *(Not required)* We will show next week that φ is linear.
(c) Show that φ is surjective.
(d) Show that φ is injective.