MATH 23a, FALL 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm Exam (take-home portion)
Due: 11 A.M. on November 22, 2004

Directions: For this exam, you will work in teams of between four and six students, which should be confirmed with the instructor before beginning the exam. (For teams with more than four students, there will be more information coming soon!)

You have until 11 A. M. on Monday, November 22, to complete this exam, at which time it should be submitted in class. You may use your own class notes (or those of your teammates), your own homework assignments (or those of your teammates), and the course textbooks (and supplemental bibliography) as your only aids. You may not use any internet resources except for the course website and the posted homework and exam solutions.

You may discuss the exam only with your teammates, and all questions should be directed only to the instructor. Obviously, the course will be continuing, and you should be going to section, so you will be able to speak with the Math 23a CA's, but you should not put them in an awkward position by asking questions directly related to the take-home, and they will be instructed to report anyone attempting to gain unfair advantage.

Each team should submit one solution set, with each team member hand-writing a solution to one of the problems. There is partial credit, but only for intelligible work. One point per problem will be awarded for *neatness only*, and one point will be awarded for *mathematical style only*, including brevity. Make sure your name and your teammates' names are prominently displayed on your work, and *please* staple your final pages together into one stack.

You may quote results from class and/or your notes with an appropriate reference, and you must cite anything you take from a textbook. Otherwise, all work should be that of you and your team.

1. **The Second Isomorphism Theorem**

   Prove the Second Isomorphism Theorem, which states:

   If V and W are subspaces of a vector space U, then

   $$V/(V \cap W) \cong (V + W)/W.$$

2. **A Finite Field**

   Consider the field $F = \mathbb{Z}/3\mathbb{Z}$ with its elements identified as 0, 1, and 2. (Properly speaking, these are representatives of equivalence classes, but we will allow the simpler notation.)

   Now define $F[x] = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid n \in \mathbb{N}, a_i \in F, \forall i\}$ to be the vector space of all polynomials with coefficients in $F = \mathbb{Z}/3\mathbb{Z}$, where addition and scalar multiplication are defined as usual. Note, however, that we also know how to multiply two polynomials, and that with this notion of multiplication, $F[x]$ actually satisfies the axioms for a commutative ring with identity.

   Let $p(x) = x^2 + 1$ be a fixed vector (polynomial) in $F[x]$, and define $I = \{a(x)p(x) \mid a(x) \in F[x]\}$ to be the subspace of $F[x]$ consisting of all multiples (polynomial multiples, not just scalar multiples) of this single vector.

   (a) Show that I is a subspace of vector space $F[x]$. (In fact, it is a *subring*, but we are only concerned with vector space properties in this part of the question.)

   (b) Define the quotient space $F[x]/I$ in terms of the data above, and find a minimal complete set of coset representatives. (Note that this is not the same as finding a basis. Since F is finite, it is possible to list all the elements of $F[x]/I$. Hint: You might consider long division of polynomials to help you classify the cosets.)

   (c) We have already seen in general that the quotient space has the structure of a vector space (so that we already have addition and scalar multiplication). Show that the natural definition of multiplication is well-defined for elements of the quotient space $F[x]/I$.

   (d) With the multiplication from part (c) and the representatives from part (b), show that $F[x]/I$ satisfies Axiom M3 and M4 (multiplicative identity and inverses) for a field by explicitly naming the identity and all multiplicative inverses. (This shows that $F[x]/I$ is a field because Axioms M1, M2, and D are inherited from the stucture of $F[x]$.)

3. Nilpotent Linear Transformations

Let $V = \mathbb{R}^n$, and let $L : V \to V$ be linear. We say that L is *nilpotent* if there is some $m \in \mathbb{N}$ such that $L^m = 0$, and we say that m is the *degree* of nilpotency if $L^m = 0$ but $L^{(m-1)} \neq 0$.

(a) Let $L : V \to V$ be nilpotent of degree m. Let $\mathbf{v} \in V$ be such that $L^{(m-1)}\mathbf{v} \neq \mathbf{0}$. Show that $\{\mathbf{v}, L(\mathbf{v}), L^2(\mathbf{v}), \ldots, L^{(m-1)}(\mathbf{v})\}$ is a linearly independent set in V. Conclude that $m \leq n$.

(b) Show that if $L : V \to V$ is nilpotent, then there is a basis for V with respect to which the matrix $A = [\alpha_{ij}]$ for L has the property that $\alpha_{ij} = 0$ whenever $i \geq j$.

(c) For the case $V = \mathbb{R}^3$, exhibit examples of nilpotent linear transformations of degrees 1, 2, and 3.

(d) Let $L : V \to V$ be nilpotent. Find all the eigenvalues of L.

(e) Show that if L is nilpotent, then $I + L$ is invertible. (If L is nilpotent, then $I + L$ is called *unipotent*.)

4. The Dual Space

Let V and W be vector spaces over the field F, and let $\mathcal{L}(V, W)$ denote the set of all linear maps from a V to W. Recall that we define addition and scalar multiplication in $\mathcal{L}(V, W)$ as follows:

$$
\begin{aligned}
(T_1 + T_2)(\mathbf{v}) &= T_1(\mathbf{v}) + T_2(\mathbf{v}) \\
(c \cdot T)(\mathbf{v}) &= c \cdot T(\mathbf{v})
\end{aligned}
$$

then $\mathcal{L}(V, W)$ is a vector space over F.

(a) Consider the case when V and W are finite-dimensional, say $\dim(V) = m$ and $\dim(W) = n$. Find a basis for $\mathcal{L}(V, W)$, and compute its dimension.

(b) In the special case (still with $dim(V) = n$) where $W = F$, we call $\mathcal{L}(V, F)$ the *dual space* of V, which we denote V^*. Since V^* is itself a vector space over F, we may then find *its* dual, $(V^*)^* = \mathcal{L}(V^*, F)$.

Show that $V \cong (V^*)^*$ by showing that the map $\varphi : V \to (V^*)^*$ is linear and injective (and hence bijective by a dimension argument), where φ is given by $\varphi(\mathbf{v}) = L_{\mathbf{v}}$, where $L_{\mathbf{v}} : V^* \to F$ is given by $L_{\mathbf{v}}(f) = f(\mathbf{v})$ for any $f \in V^*$.

(Such an isomorphism is called *natural* because it does not depend on the bases chosen for any of the vector spaces.)