6. Long Exact Sequences

Let $V_1, \ldots, V_n$ be finite-dimensional vector spaces over the same field, and suppose we have the following sequence of linear maps:

$$\{\mathbf{0}\} \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \quad \cdots \quad \longrightarrow V_n \longrightarrow \{\mathbf{0}\}$$

where $\varphi_i : V_i \longrightarrow V_{i+1}$, for $0 \leq i \leq n$.

(Here, we let $V_0 = V_{n+1} = \{\mathbf{0}\}$ for consistency in the subscripts.)

Such a collection of vector spaces and linear maps is known as a *(long) exact sequence* when they satisfy the relationship:

$$Im(\varphi_i) = Ker(\varphi_{i+1}), \quad \text{for } 0 \leq i \leq n.$$

Show that $\displaystyle\sum_{i=0}^{n} (-1)^i \cdot dim(V_i) = 0.$