Last Name: _____

First Name: _____

MATH 23b, SPRING 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm (part 1)
March 12, 2004

Directions: You have one hour (more like 53 minutes) for this exam. No calculators, notes, books, etc. are allowed. Please answer on the pages provided. Show all work!

Problem	Points	per part	Score
1	30	2 each	
2	6		
3	16	4 each	
4	12	4/8	
5	12	4/8	
6	12	4 each	
Total	88	88	

1. **True or False** For each statement, please circle either **T** or **F**.

    T or **F** Every closed set in $\mathbb{R}^n$ is compact.

    T or **F** Every bounded discrete set in $\mathbb{R}$ is finite.

    T or **F** The intersection of every nested collection of bounded sets in $\mathbb{R}^n$ is non-empty.

    T or **F** For any set $A \subset \mathbb{R}^n$, we have $\partial A = \overline{A} \setminus A^\circ$.

    T or **F** Every limit point of a set $A \subset \mathbb{R}^n$ is also an accumulation point of A.

    T or **F** Every isolated point of a set $A \subset \mathbb{R}^n$ is also a limit point of A.

    T or **F** If $A \subset \mathbb{R}^n$ is discrete, then A is not open.

    T or **F** Every set $A \subset \mathbb{R}^n$ has a non-empty boundary.

    T or **F** Every non-trivial set $A \subset \mathbb{R}^n$ has at least one accumulation point.

For the next three questions, let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in \mathbb{R}^n$.

    T or **F** All of the directional derivatives of f are continuous at $\mathbf{a}$.

    T or **F** All of the directional derivatives of f exist at $\mathbf{a}$.

    T or **F** The function f is continuous at $\mathbf{a}$.

For the remaining questions, let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous.

    T or **F** If $A \subset \mathbb{R}^n$ is open, then $f(A)$ is open in $\mathbb{R}^m$.

    T or **F** If $A \subset \mathbb{R}^n$ is closed, then $f(A)$ is closed in $\mathbb{R}^m$.

    T or **F** If $A \subset \mathbb{R}^n$ is compact, then $f(A)$ is compact in $\mathbb{R}^m$.

2. State the Bolzano-Weierstrass Theorem.

3. Let $A = \{(x, 0) \mid 0 < x < 1\}$ be considered as a subset of $\mathbb{R}^2$.

   (a) **True** of **False** with (short) explanation: A is closed.

   (b) **True** of **False** with (short) explanation: A is open.

   (c) Find (without proof) ∂A, the *boundary* of A.

   (d) Find (without proof) an open cover of A that has no finite sub-cover.

4. (a) Let $A \subset \mathbb{R}^n$. Define what it means for A to be *compact*.

   (b) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, and let $A \subset \mathbb{R}^n$ be a compact set. Show that f attains its maximum value on A. (That is, show that there is some $\mathbf{x} \in A$ such that $f(\mathbf{x}) \geq f(\mathbf{y})$ for every $\mathbf{y} \in A$.)

5. (a) Define what it means for $f : \mathbb{R}^n \to \mathbb{R}$ to be *differentiable* at $\mathbf{a} \in \mathbb{R}^n$.

   (b) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(\mathbf{x}) = ||\mathbf{x}||$ is continuous at every $\mathbf{x} \in \mathbb{R}^2$ but not differentiable at $\mathbf{0} \in \mathbb{R}^2$.

6. Let $f : \mathbb{R}^4 \to \mathbb{R}$ be given by

$$f(x, y, z, w) = e^{xy} + xyz - z \sin w,$$

and let $\mathbf{a} = (2, 0, 1, \frac{\pi}{2})$.

(a) Find $\nabla f(\mathbf{a})$.

(b) Find an expression for the tangent space to f at $\mathbf{a}$.
   (Your expression should have the form $T(x, y, z, w) = \cdots$)

(c) Find the directional derivative, $(D_{\mathbf{h}} f)(\mathbf{a})$, for the direction vector
   $\mathbf{h} = (-3, 1, 2, -1)$.