MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS

In class on Wednesday (2/9), we rushed the end of the proof of the following theorem, and so here is a cleaned up version.

Theorem. A set $A \subset \mathbb{R}^n$ is closed if and only if it contains all of its limit points.

Proof:

($\Rightarrow$) Let A be closed, and suppose x is a limit point of A. (We need to show that $x \in A$.) Since A is closed, by definition, A^c is open.

If x were in A^c, then there would be some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A^c$. But this implies that if $\{x_n\}$ were a sequence converging to x, then there would be some $N \in \mathbb{N}$ such that $x_n \in A^c$ whenever $n > N$, and hence this sequence would not be entirely within A. In other words, there would be no sequence in A that converged to x, and hence x would not be a limit point of A.

Hence x cannot be in A^c and instead must be in A, which is what we wanted to show.

($\Leftarrow$) Suppose A contains all of its limit points. (We need to show that A^c is open.)

Take $x \in A^c$. We would like to show that there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A^c$.

Consider the particular value $\varepsilon = 1$. If $B_1(x) \subset A^c$, then we are done. If not, then take $x_1 \in A \cap B_1(x)$. Now consider $\varepsilon = \frac{1}{2}$. If $B_{1/2}(x) \subset A^c$, then we are done. If not, then take $x_2 \in A \cap B_{1/2}(x)$. Now consider $\varepsilon = \frac{1}{3}$, and continue in this manner. If at any time, we produce an $\varepsilon = \frac{1}{n}$ such that $B_{1/n}(x) \subset A^c$, then we are done.

If not, then we will have produced a sequence $\{x_n\}$ that consists of points in A that converge to x, and hence x would be a limit point of A that was not contained in A, which contradicts our hypothesis. Hence there must have been some $\varepsilon = \frac{1}{n}$ such that $B_{1/n}(x) \subset A^c$, which is what we wanted to show.