Math 23a Soltuion: Problem C

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(a) We define $\mathbb{Q} = \{(a, b) | a \in \mathbb{Z}, b \in \mathbb{N}\}/\sim$, where $\sim$ is defined by $(a, b) \sim (c, d)$ if and only if $ac = bd$. This rule is of course just the "cross multiplication" of fractions you learned back in fourth grade. Note that I stipulated that b, the denominator, is a natural number. This is not necessary; all one has to do is make sure that $b \neq 0$. But for any rational number, we can always represent it by a fraction with a positive denominator, simply by multiplying the numerator and denominator by -1 if the deonimnator is negative. Since it also makes the proofs here a little easier, we will just assume that $b > 0$, although this choice is not ideal for proving other things, such as multplicative inverses.
(b) Motivated by what we already know these operations have to look like, we define

$$[(a, b)] + [(c, d)] = [(ac + bd, bd)],$$
$$[(a, b)] \cdot [(c, d)] = [(ac, bd)], \text{ and}$$
$$[(a, b)] < [(c, d)] \text{ if and only if } ad < bc.$$

In the definition of $<$, we simply cleared the denominators. Note that if we allowed denominators to be negative, we would have to be more careful, since if b and d had different signs, then clearing the denominators by multiplying through by bd would reverse the direction of the inequality, and we would have to do a case by case analysis, or else basically do what we did here and ignore negative denominators.
(c) We first prove that $[(0, 1)]$ is an additive identity on $\mathbb{Q}$:

$$[(a, b)] + [(0, 1)] = [(1 \cdot a + 0 \cdot b, b \cdot 1)] = [(a, b)].$$

Now for any element $[(a, b)]$, we compute

$$[(a, b)] + [(-a, b)] = [(ab - ba, b^2)] = [(0, b^2)] = [(0, 1)]$$

where the last equality follows since $b^2 \cdot 0 = 1 \cdot 0$. Thus $[(a, b)]$ has additive inverse $[(-a, b)]$.
(d) According to our definitions, $[(0, 1)] < [(a, b)]$ if and only if $a > 0$. Thus, if $[(a, b)] > 0$ and $[(c, d)] > 0$, then $a > 0$ and $c > 0$. Note also that by assumption b and d are also positive integers. Then $[(a, b)] + [(c, d)] = [(ad + bc, bd)]$ and $ad + bc > 0$ since the positive integers are closed under integer addition and multiplication. Thus $[(a, b)] + [(c, d)] > [(0, 1)]$ and the positive rationals are closed under addition.