SOLUTION SET 2A

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Math 23a
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2. Using the uniqueness of additive inverses in a ring (see HW #1.8), prove that $(-a)(-b) = ab$. *(Hint: Show that each of the two expressions is the additive inverse of some third expression.)*

We claim that $(-a)b$ is the additive inverse of both $(-a)(-b)$ and ab. First we prove that $(-a)b$ is the additive inverse of $(-a)(-b)$, or $(-a)(-b) + (-a)(b) = 0$:

$$(-a)(-b) + (-a)(b) = (-a)((-b) + b) = (-a)0 = 0.$$

Similarly, we prove that $(-a)b$ is the additive inverse of ab:

$$ab + (-a)b = (a + (-a))b = 0b = 0.$$

By the uniqueness of additive inverses in a ring, we conclude that $(-a)(-b) = ab$. In this proof, be careful with expressions like $-ab$. Make it clear whether we are talking about $(-a)b$, i.e. the product of the additive inverse of a and b, or $-(ab)$, the additive inverse of the product of a and b. They are of course expressions for the same ring element, but you can't take it for granted in a basic proof like this where you are expected to make use only of the ring axioms.

4. If V is a vector space and $\mathbf{v} \in V$, show that $(-1)\mathbf{v} = -\mathbf{v}$.
(Note that there really is something to do here. The expression on the right is the additive inverse of the vector, and the one on the left is the vector multiplied by the scalar -1, and these are not *a priori* the same thing.)

The main idea to this proof is that the distributivity axioms for vector spaces relate the vector addition operations and the field addition operations to each other in a natural way. The basic fact to use is that $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$. Here is a quick proof: We can write $0\mathbf{v} + \mathbf{v} = 0\mathbf{v} + 1\mathbf{v} = (0 + 1)\mathbf{v} = 1\mathbf{v} = \mathbf{v}$, so we have $0\mathbf{v} + \mathbf{v} = \mathbf{v}$. Adding the additive inverse of $\mathbf{v}$ to both sides gives $0\mathbf{v} = \mathbf{0}$.

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Method 1: Using this fact, we can proceed as follows:

$$\begin{aligned}(-1)\mathbf{v} &= (-1)\mathbf{v} + \mathbf{0} \\ &= (-1)\mathbf{v} + (\mathbf{v} + (-\mathbf{v})) \\ &= (-1)\mathbf{v} + (1\mathbf{v} + (-\mathbf{v})) \\ &= ((-1)\mathbf{v} + 1\mathbf{v}) + (-\mathbf{v}) \\ &= ((-1) + 1)\mathbf{v} + (-\mathbf{v}) \\ &= 0\mathbf{v} + (-\mathbf{v}) \\ &= \mathbf{0} + (-\mathbf{v}) \\ &= -\mathbf{v}\end{aligned}$$

Method 2: As in problem 2, show that both $(-1)\mathbf{v}$ and $-\mathbf{v}$ are additive inverses of $\mathbf{v}$. This is also a valid approach, because it is a fact that additive inverses are unique in a vector space, just as additive inverses are unique in a ring. Here is a quick proof: Let $\mathbf{u} \in V$. Suppose $\mathbf{u} + \mathbf{v} = \mathbf{0}$ and $\mathbf{u} + \mathbf{w} = \mathbf{0}$. Then $\mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v} + (\mathbf{u} + \mathbf{w}) = (\mathbf{v} + \mathbf{u}) + \mathbf{w} = \mathbf{0} + \mathbf{w} = \mathbf{w}$. Now, both $(-1)\mathbf{v}$ and $-\mathbf{v}$ are additive inverses of $\mathbf{v}$:

$$\mathbf{v} + (-1)\mathbf{v} = 1\mathbf{v} + (-1)\mathbf{v} = (1 + (-1))\mathbf{v} = 0\mathbf{v} = \mathbf{0},$$

and $-\mathbf{v}$ is simply the expression for the additive inverse of $\mathbf{v}$. Hence we have $(-1)\mathbf{v} = -\mathbf{v}$.

The proofs shown above appeal only to axioms that are common to all vector spaces. On the other hand, writing $\mathbf{v} = (v_1, v_2, ..., v_n)$ and using the usual rules for $\mathbb{R}^n$ to show $(-1)\mathbf{v} = -\mathbf{v}$ can illustrate how the above proofs apply to the specific instance of the vector space $\mathbb{R}^n$, but does not give a general proof that works for vector spaces with different rules of vector addition and scalar multiplication. If you wrote the proof only for $\mathbb{R}^n$, then look at the steps you have written and think about what vector space axioms you were making use in each step.