Math 23a Soltuion: Problem D

David Jackson-Hanen

October 14, 2004

For $c \in \mathbb{R}$, we defined

$$V(c) = \{f : [0,1] \to \mathbb{R} | f \text{ is continuous and } f(0) = c\}.$$

To determine whether $V(c)$ is a subspace of $C[0,1]$, we must check whether it is closed under addition and scalar multiplication (this is theorem 3.2.2 of your linear algebra book). In fact, we can be very clever and just check the following: if $f(x)$ and $g(x)$ are in $V(c)$, and a, b are any real numbers, then $af(x) + bg(x) \in V(c)$. This one fact suffices because, were we to prove it, we could simply take $a = b = 1$ to prove closure under addition, and then take $b = 0$ to prove closure under scalar multiplication.

Now assume $f(x), g(x) \in V(c)$ and $a, b \in \mathbb{R}$. By definition, $af(x) + bg(x) \in V(c)$ iff $af(0) + bg(0) = c$, or $ac + bc = c$, $(a + b)c = c$. Now, this equation has to be true for any a and b, and the only way to ensure that is to have $c = 0$ (otherwise we could divide both sides by c and conclude $a + b = 1$, which is definitely not true for all real numbers!). Thus, $V(c)$ is a vector space iff $c = 0$.