SOLUTION SET 2E

ALEX LEVIN

Math 23a
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8. a. To prove that ℓ^2 is a subspace of V, we need only show that it is closed under addition and scalar multiplication. So, let $a = (a_0, a_1, a_2, \ldots)$, $b = (b_0, b_1, b_2, \ldots) \in \ell^2$, and let $\lambda \in \mathbb{R}$. Clearly, $a_i + b_i \in \mathbb{R} \forall i$, $\lambda a_i \in \mathbb{R} \forall i$.

We now prove that $a + b \in \ell^2$, i.e. that $\sum_{i=1}^{\infty} |a_i + b_i|^2$ converges. We know that $\sum |a_i|^2$ and $\sum |b_i|^2$ converge because $a, b \in \ell^2$ (we have dropped the range of the sums because it is clear from context).

Now note that by the triangle inequality, we have $|a_i + b_i| \leq |a_i| + |b_i|$, and so

$$0 \leq \sum |a_i + b_i|^2 \leq \sum (|a_i| + |b_i|)^2 = \sum \left(|a_i|^2 + 2|a_i||b_i| + |b_i|^2 \right).$$

Now, this last expression is just equal to $\sum |a_i|^2 + \sum |b_i|^2 + \sum |a_i||b_i|$. We know that the first two series converge by definition, and we will be done if we can demonstrate that the third one converges.

There are two good ways of doing this. The first is to notice that for any positive reals x and y, $(x - y)^2 \geq 0$, but $(x - y)^2 = x^2 + y^2 - 2xy \geq 0$, so $xy \leq 2xy \leq x^2 + y^2$. We can now apply this to the $|a_i||b_i|$ terms, to see that every term of the infinite sum $\sum |a_i||b_i|$ is less than or equal to the corresponding term in a converging series $\sum (|a_i|^2 + |b_i|^2)$, so $\sum |a_i||b_i|$ must converge.

Another way of demonstrating this is to notice that either $|a_i||b_i| \leq a_i^2$, or $|a_i||b_i| \leq b_i^2$, depending on whichever of a_i or b_i is greater. But in both cases, $|a_i||b_i| \leq |a_i^2| + |b_i^2|$, and we have again compared the series of cross terms to a converging series. In any case, we now see that $\sum |a_i + b_i|^2$ converges.

Proving closure under scalar multiplication is a lot easier. For given $a \in \ell^2$, $a = (a_0, a_1, a_2, \ldots)$, $ca = (ca_0, ca_1, ca_2, \ldots)$, and $\sum |ca_n|^2 = \sum c^2 |a_n|^2 = c^2 \sum |a_n|^2$, which converges because $\sum |a_n|^2$ converges.

b. To prove that $\ell^1 \subset \ell^2$, we must demonstrate that $\forall a \in \ell^1$, $a \in \ell^2$. Equivalently, if $a = (a_0, a_1, \ldots)$ and $\sum |a_n|$ converges, we must show that $\sum |a_n|^2$ converges. Notice that since $\sum |a_i|$ converges, its terms have to tend to 0, and in particular, only finitely many of the a_i's will be greater than or equal to 1. Let $N \in \mathbb{N}$ be a number such that $|a_i| < 1 \ \forall i > N$.

Now let us consider $\sum |a_n|^2 = \sum_{n=1}^{N} |a_n|^2 + \sum_{n=N+1}^{\infty} |a_n|^2$. The former of these, being a finite sum, converges. As for the latter, for all $n > N$, we have $|a_n| > |a_n|^2$, so, since $\sum_{n=N+1}^{\infty} |a_n|$ converges, so too must $\sum_{n=n+1}^{\infty} |a_n|^2$, by the comparison test.

Another way to look at this problem is to let $S = \sum |a_n|$. Then,

$$\infty > S^2 = \left(\sum |a_n|\right)^2 = \sum |a_n|^2 + \text{positive cross terms} \geq \sum |a_n|^2 \geq 0.$$

So, $\sum |a_n|^2$ converges.

Finally, $\ell^1 \neq \ell^2$, because we can take $a = (1, 1/2, 1/3, \ldots)$. Then $a \in \ell^2$ since $1 + (1/2)^2 + (1/3)^2 + \ldots = \pi^2/6$ by a classical result in analysis. However, the harmonic series $1 + 1/2 + 1/3 + \ldots$ diverges, and so $a \notin \ell^1$