Solution Set 6B

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Math 23a
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4 (B) Let $V = \mathbb{R}^n$, and let $\mathbf{u}, \mathbf{v} \in V$. If $A : V \longrightarrow V$ is (the matrix for) a linear transformation, then define the following bilinear form:

$$f_a(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t A\mathbf{v}.$$

(a) Show that f_a is indeed a bilinear form.

Solution: Take $c \in F$. Note that $f_a(c\mathbf{u}, \mathbf{v}) = (c\mathbf{u}^t)A\mathbf{v} = c(\mathbf{u}^t A\mathbf{v}) = cf_a(\mathbf{u}, \mathbf{v})$. Likewise, $f_a(\mathbf{u}, c\mathbf{v}) = \mathbf{u}^t A(c\mathbf{v}) = c(\mathbf{u}^t A\mathbf{v}) = cf_a(\mathbf{u}, \mathbf{v})$. Notice that we have used the linearity of matrices to pull c out of our expression. Similarly,

$$f_a(\mathbf{u}+\mathbf{w}, \mathbf{v}) = (\mathbf{u}+\mathbf{w})^t A\mathbf{v} = (\mathbf{w}^t + \mathbf{u}^t)A\mathbf{v} = \mathbf{w}^t A\mathbf{v} + \mathbf{u}^t A\mathbf{v} = f_a(\mathbf{u}, \mathbf{v}) + f_a(\mathbf{w}, \mathbf{v}).$$

Finally, $f_a(\mathbf{u}, \mathbf{v} + \mathbf{w}) = \mathbf{u}^t A(\mathbf{v} + \mathbf{w}) = \mathbf{u}^t A\mathbf{v} + \mathbf{u}^t A\mathbf{w} = f_a(\mathbf{u}, \mathbf{v}) + f_a(\mathbf{u}, \mathbf{w})$.
Hence, f_a is a bilinear form.

(b) Give a necessary and sufficient condition on the matrix A that makes f_a alternating.

Solution: I claim that f_a is alternating if and only if $A = -A^t$.

Consider $f_a(\mathbf{v}, \mathbf{v})$, where $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$. Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$.

Then

$$\begin{aligned} \mathbf{v}^t A\mathbf{v} &= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} v_1 a_{11} + \cdots + v_n a_{1n} \\ \vdots \\ v_1 a_{n1} + \cdots + v_n a_{nn} \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= v_1(v_1a_{11} + ... + v_na_{nn}) + ... + v_n(v_1a_{n1} + ... + v_na_{nn}) \\
&= v_1^2a_{11} + \cdots + a_n^2a_{nn} + v_1v_2(a_{12} + a_{21}) + \cdots + v_1v_n(a_{1n} + a_{n1}) + \\
&\quad + v_2v_3(a_{23} + b_{32}) + \cdots + v_{n-1}v_n(a_{(n-1)n} + a_{n(n-1)}) \\
&= \left(\sum_{j=1}^{n} \sum_{i=1}^{n} v_jv_i(a_{ji} + a_{ij}) \right) /2. \quad Eq.(1).
\end{aligned}
$$

Clearly, then, if $A = -A^t$, and therefore $a_{ij} = -a_{ji}$, then $Eq.(1)$ reduces to 0. Hence, $A = -A^t$ is certainly a sufficient condition. Moreover, for f_a to be alternating, $(Eq.1)$ must hold for all $\mathbf{v}$. Hence, define $\mathbf{v}_{ij} = (0, \ldots, 1, 0, \ldots, 1, 0, \ldots, 0)$, namely a vector with all 0's except for 1's in the i-th and j-th position. With respect to $\mathbf{v}_{ij}$ equation 1 simplifies to just $a_{ij} + a_{ji} = 0$. Hence, we need $a_{ij} + a_{ji} = 0$ as well for $1 \le i, j \le n$. But this is equivalent to requiring that $a_{ij} = -a_{ji}$ for all of these i, j , or in other words, that $A = -A^t$. Thus, our condition is necessary as well.

Notes: On the whole, I was quite impressed with the quality of submitted assignments. I do have a few notes, however:

1. Just remember that it's important to prove all assertions. Many people lost points on what must have seemed like fairly picky grading; some people, for example, concluded at an equation similar to what I called $Eq.$ (1) by saying that since the whole sum has to equal 0, each of the terms must equal 0. This is not a rigorous argument - it's entirely possible (although in this case, ultimately untrue), that terms could cancel nicely and you would not need every single co-efficient to be 0. A somewhat more detailed version of this argument proceeded by asserting that the v_i could be anything, so we need to make the co-efficients 0 - while this is getting at the point, it's still not quite thorough enough. The clearest proofs proceeded as above by actually plugging in vectors to isolate co-efficients.

2. Additionally, many people claimed that since $Eq.$ (1) represented a linear combination of multinomials, the fact that these multinomials were linearly independent implied that their combination must be trivial since it equaled 0, and hence all the co-efficients must equal 0. While this may be true, it's certainly nothing that we discussed in class, so full credit was only given to students who actually proved this assertion.

3. Moreover, it's not sufficient to just examine the $n = 2$ case and say that it generalizes nicely. With some detailed explanation, you could probably prove that your condition is necessary by examining $n = 2$ (although this is not preferable), but sufficiency for any n is more problematic. At any rate, simply writing down formulas for $n = 2$ and $n = 3$, and then saying that the idea in the general case is the same is not a strong proof.

4. This is really picky, but the question did ask to find a necessary *and* sufficient condition. While proving that $A = -A^t$ is necessary is probably harder than showing that it's sufficient, you should still show both sides of the argument. Many people presented excellent proofs that their condition was necessary, but didn't mention sufficiency at all.

5. Finally, be sure not to mix up the requirements for a form to be skew-symmetric versus alternating. A number of people attempted to prove that their condition was sufficient by concluding that it implied that $f_a(\mathbf{u},\mathbf{v}) = -f_a(\mathbf{v},\mathbf{u})$. Since we are over a real field, skew-symmetry does in fact imply that f_a is alternating, but you need to at least say that - just proving skew-symmetry with no additional explanation is insufficient.

5 (5B) Show that not every skew-symmetric multilinear form $f : V^n \longrightarrow F$ is alternating by constructing an example.

Solution: Let $V = F = \mathbb{Z}/2\mathbb{Z}$, and define $f : V^n \longrightarrow F$, where $f(a_1, \ldots, a_n) = a_1 a_2 \cdots a_n$. Consider $c \in F$. Note that for any a_i, $f(a_1, \ldots, ca_i, \ldots, a_n) = a_1 \cdots ca_i \cdots a_n = c(a_1 \cdots a_n) = cf(a_1, \ldots, a_n)$. Likewise, for any $a_i, w \in V$,

$$\begin{aligned} f(a_1, \ldots, a_i + w, \ldots, a_n) &= a_1 \cdots (a_i + w) \cdots a_n \\ &= a_1 \cdots a_i \cdots a_n + a_1 \cdots w \cdots a_n \\ &= f(a_1, \ldots, a_i, \ldots, a_n) + f(a_1, \ldots, w, \ldots, a_n). \end{aligned}$$

Hence, f is a multilinear form.
Moreover, since $1 \equiv -1 (\text{mod } 2)$,

$$\begin{aligned} f(a_1, \ldots, u, \ldots, v, \ldots, a_n) &= a_1 \cdots u \cdots v \cdots a_n \\ &= -(a_1 \cdots u \cdots v \cdots a_n) \\ &= -(a_1 \cdots v \cdots u \cdots a_n) \\ &= -f(a_1, \ldots, v, \ldots, u, \ldots, a_n). \end{aligned}$$

Hence, f is skew-symmetric as well. However, as $f(1, 1, \ldots, 1) = 1 * 1 * \cdots * 1 = 1 \neq 0$, f is not alternating. Thus, not every skew-symmetric multi-linear form is alternating.

Notes: A few quick comments for this question:

1. A number of proofs proceeded by analyzing the proof that any alternating, multilinear form is skew-symmetric, and concluded that, in the reverse direction, the fact that $2 \equiv 0$ (*mod* 2) was problematic. While this is true, this does not prove that not every skew-symmetric multilinear form is alternating - just because that particular proof does not work does not necessarily show that there is no other proof out there. The best way to definitely show this fact is to construct a counter-example, as asked.

2. This problem was, to a great degree, about thoroughness. Remember to check that your form really is skew-symmetric and multilinear. A number of people failed to adequately demonstrate skew-symmetry. In particular, remember that you need to show that the $form$ is skew-symmetric. Hence, picking two vectors and showing that the form behaves appropriately with respect to just those two is not sufficient - if it were, every multilinear form would in fact be skew-symmetric because you could switch the 0 vector with any other vector and pull out a negative sign as $-0 = 0$.

3. Just a reminder that for a question such as this one, many proofs were acceptable. Since you only have to find one counter example, constructing a form specific to the $n = 2$ case was fine, as was considering the case where V was a 1 dimensional vector space. I chose to write my proof for any n simply because I wanted to.

3. Finally, and again this is a bit picky, you really do need to demonstrate by example. Many people produced excellent multilinear, skew-symmetric, but just stated that their form was not alternating without proof. You need to demonstrate that you understand the alternating condition by actually plugging in a vector. Along these lines, some people tried to show that their form was not alternating with respect to *all* non-zero vectors. While this is admirable (and true sometimes - for example, in my solution), it's unnecessary - you only need one counter-example to demonstrate that your function is not alternating.

4. Overall, this proof (questions 4 and 5) was really long, and thus graded out of 25 points.