SOLUTION SET 8A

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2. Recall homework problem # 6.4 in which you showed that for $V = \mathbb{R}^n$ and $u, v \in V$, if $A : V \to V$ is a linear transformation, then

$$f_A(u, v) = u^t A v \tag{1}$$

defines a bilinear form.

Give a necessary and sufficient condition on A that makes f_A an inner product. (Full points for a complete answer in the $n = 2$ case.)

For $f_A(\cdot, \cdot)$ to be an inner product, it must be bilinear, symmetric, and positive definite. We consider each of these conditions in turn.

Bilinear: From homework problem # 6.4, any A will make f_A bilinear.

Symmetric: We can actually do this quite elegantly for the general $(n \times n)$ case; several students have figured this out. We show that $A = A^t$ is a necessary and sufficient condition.

To prove that it is necessary, let

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix} \tag{2}$$

Then, imposing

$$f_A(e_i, e_j) = f_A(e_j, e_i) \tag{3}$$

$$\implies e_i^t A e_j = e_j^t A e_i \tag{4}$$

we find by a simple computation that $a_{i,j} = a_{j,i}$ for all i and j, i.e. A is symmetric ($A = A^t$). In the above equations e_i is the column vector with a 1 in the i^{th} row and 0's everywhere else.

To prove that $A = A^t$ is sufficient, notice that since $f_A(u, v) = u^t A v$ is a scalar, it is equal to its own transpose, but

$$(u^t A v)^t = v^t A^t u \tag{5}$$

by a standard property of transposes. Then,

$$f_A(v, u) = v^t A u \tag{6}$$

so if $A = A^t$, (5) and (6) tell us that $f_A(u, v) = f_A(v, u)$ as desired.

Date: December 20, 2004.

Alternately, we can do it directly for the $n = 2$ case; let $u = \binom{u_1}{u_2}$ and $v = \binom{v_1}{v_2}$. This gives us

$$f_A(u,v) = av_1u_1 + bv_2u_1 + cu_2v_1 + du_2v_2 \tag{7}$$
$$f_A(v,u) = au_1v_1 + bv_1u_2 + cv_2u_1 + du_2v_2 \tag{8}$$

For symmetry to hold, we must have

$$b(v_2u_1 - v_1u_2) = c(v_2u_1 - u_2v_1) \tag{9}$$

Since this must hold for any choice of u_1, u_2, v_1, v_2, namely for $u = \binom{1}{0}$ and $v = \binom{0}{1}$, the necessary requirement is that $b = c$. And, retracing our steps through the computations we see that it is also sufficient.

Remark 1. *For some reason, way too many people were confused about the difference between necessary and sufficient conditions. Upon reaching (9), they just claimed that $b = c$, and did not even talk about necessity and sufficiency as distinct things to check (and certainly did not pick particular values for u and v). I wrote the following comment on these papers: "What if $u_1v_2 = u_2v_1$?" Please be careful about the difference.*

Positive Definite: Let us do this in the $n = 2$ case. Suppose

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \tag{10}$$

(we know that A has to be symmetric by the above discussion) and $u = \binom{x}{y}$. We need to show that $f_A(u,u) \geq 0 \forall u$ and that $f_A(u,u) = 0 \iff u = 0$. Now,

$$\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + dy^2 \tag{11}$$

If $y = 0$, but $x \neq 0$, our form must evaluate to something greater than 0. Hence, $ax^2 > 0$, which means that $a > 0$. If $y \neq 0$, then we can divide the right hand side of (11) by y^2 to obtain

$$a\left(\frac{x}{y}\right)^2 + 2n\left(\frac{x}{y}\right) + d \tag{12}$$

which must be strictly greater than 0. Notice that this is just a quadratic equation in x/y that is concave up (as $a > 0$ is a necessary condition, by the discussion above). We will get the result as long as it has no real roots. This will happen if the discriminant Δ is strictly less than 0. But $\Delta = 4(b^2 - ad)$, so $\Delta < 0 \Longrightarrow ad - b^2 > 0$, which means that $\det A > 0$ (notice that this also means that $d > 0$). Clearly, $f_A(0,0) = 0$. So, we've found our necessary conditions.
They are sufficient because given $u = \binom{x}{y}$, if $y = 0$, $a > 0$ implies that $f_A(u,u) \geq 0$ and is equal only when $x = 0$ as well. For $y \neq 0$ the fact that the discriminant to our quadratic is less than 0 means that it has no real roots, which (together with $a > 0$) means positivity is satisfied.

The conditions are thus (in the $n = 2$ case): A is a symmetric matrix with positive diagonal entries, and positive determinant.