Math 23b Solution: Problem D

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(a) Since B is open, ∂B is the set of all limit points of B that are not themselves in B; $\partial B = \overline{B}/B^\circ = \overline{B}/B$. Hence to prove that $\partial B = [0,1]/B$, it suffices to show that $\overline{B} = [0,1]$. For one direction, note that we defined B such that $B \subset [0,1]$ (actually we have not quite rigorously defined B such that this is the case; to ensure this, we would have to be a bit careful in choosing the original enuremation of A. However, in this problem we are cleary to assume that the enumeration of A has been chosen such that $B \subset [0,1]$), and since $[0,1]$ is closed this implies $\overline{B} \subset [0,1]$. For the other direction, we note that B contains the set $\mathbb{Q} \cap (0,1)$, which is dense in $[0,1]$, and so $[0,1] \subset \overline{B}$, hence $[0,1] = \overline{B}$.
(b) Assume ∂B has measure zero. Then we can cover ∂B with a countable collection of open set $C = \{C_i\}$ such that

$$\sum_{i=1}^{\infty} v(C_i) < \frac{1}{4}$$

Since B, by construction, can be covered by a countable collection of open intervals the sum of whose volume is $\frac{1}{2}$, and by (a) $[0,1] = B \cup \partial B$, this implies that $[0,1]$ can be covered by a countable collection of open intervals the sum of whose lengths is less than $\frac{3}{4}$. Intuitively, this is not possible; since $[0,1]$ has "length" one, any open cover should sum to more than one. If this argument satisfies you, you can move on to the next section. In the remainder of this section, we will give a rigorous proof that $[0,1]$ cannot be covered by a countable collection of open intervals the sum of whose lengths is less than one.

It will be convenient to prove a slightly more general fact: that any countable cover of the interval $[a,b]$ is such that the sum of the lengths of the intervals is greater than $b-a$. Since closed intervals are compact, all countable covers have finite subcovers; since taking finite subcovers will only decrease the total length of the cover, it suffices to prove the theorem for finite covers of the interval $[a,b]$. We prove the theorem for finite covers by induction on n, the number of intervals in the cover. For $n = 1$, the cover must consist of a single set (c,d) with $[a,b] \subset (c,d)$. For this to be the case, we must have $c < a$ and $b < d$, and so $b - a < d - c$.

Now assume the theorem holds for all covers with n open intervals of all closed intervals; we will show that it also holds for all covers with $n+1$ intervals. Let $[a, b]$ be an interval, and let A be an open cover by $n+1$ open intervals. Then there exists some $B \in A$ with $a \in B$. If $[a, b] \subset B$ then $b - a < v(b)$ and we are done. If not, then since B is an open interval there exits a unique $c \in (a, b)$ with $[a, c) \subset B$ and $B \cap [c, d] = \emptyset$. Then $A/\{B\}$ is an open cover of $[c, d]$ by n open intervals. By induction, the sum of the lengths of these intervals must be at least $b - c$ (this is where we need the induction to apply to all closed intervals). Since B must have length greater than $c - a$, the sum of the lengths of the elements of intervals of A is

$$v(b) + \sum_{C \in A/\{B\}} v(C) > (c - a) + (b - c) = b - a$$

This proves the theorem.

(c) We recall that a function is integrable iff it is continuous except for possibly a set of measure zero. Now the characteristic function of a set is discontinuous precisely at the boundary of that set; thus χ_B will be integrable iff ∂B has measure zero. But by part (b) ∂B does not have measure zero, and so χ_B is not integrable.