SOLUTION SET 1B

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MATH23B
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5. Define $f : [0,1] \longrightarrow \mathbb{R}$ as follows (note that $f(0) = 1$):

(1)
$$f(x) = \begin{cases} 0, & \textbf{if } x \notin \mathbb{Q} \\ \frac{1}{q}, & \textbf{if } x \in \mathbb{Q} \textbf{ and } x = \frac{p}{q} \textbf{ in lowest terms} \end{cases}$$

(a) Graph f
Almost all of you got this one right.

(b) Show that f is not continuous at any rational x.
Let $x \in \mathbb{Q}$, $x = p/q$ in lowest terms. Notice that if $x \neq 1$, for large enough M, $x + \pi/n \in [0,1]$ for all $n > M$. Consider the sequence $\{x_n\}$ where

(2)
$$x_n = \begin{cases} x, & \text{if } n \leq M \\ x + \pi/n, & \text{if } n > M \end{cases}$$

Notice that $x + \pi/n \notin \mathbb{Q}$ for all n since if $x + \pi/n = r \in \mathbb{Q}$, $r - x = \pi/n \in \mathbb{Q}$, which is not the case. Thus, the tail of $\{x_n\}$ is a sequence of irrationals, and so, $f(x_n) \longrightarrow 0$. But $x_n \longrightarrow x$, while $f(x) = 1/q \neq 0$, (if $x = 0$, $f(x) = 1$) so we've constructed a sequence such that $x_n \longrightarrow x$ though $f(x_n)$ does not converge to $f(x)$. Thus, f is not continuous at x. (If $x = 1$, we can redo the argument with the sequence $1 - \pi/n$.)

(c) Show that f is continuous at any irrational x Choose $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Consider the set $S \subset \mathbb{Q}$ of rational numbers y such that when y is written as p/q in lowest terms, $q < N$ (so that possibly $f(y) > \epsilon$). Clearly S is a finite set, so we can choose $\delta < \min\{|x - y|\}_{y \in S}$ (because S is finite, the minimum is well-defined). Given $x' \in B_\delta(x)$, if x' is irrational, $f(x') = f(x) = 0$. If $x' \in \mathbb{Q}$, then, when it is written in lowest terms as p/q. q must be greater than N (all the rationals with denominator smaller than N are farther than δ away from x by construction) so $|f(x') - f(x)| < 1/N < \epsilon$. In any case, we have established continuity at $x \notin \mathbb{Q}$.

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