Math 23b Solution: Problem D

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We have $A = \{(a,b) \in \mathbb{R}^2 \mid a, b \in \mathbb{Q}\}$ and $B = \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 < 1\}$.

(a) $A^\circ = \emptyset$. To see this, pick $\epsilon > 0$, and $(a,b) \in A$. Let a' be an irrational number such that $|a - a'| < \epsilon$, which we know exists by, say, taking $a' = a + \pi/n$ for n sufficiently large. Then $(a', b) \notin A$ and $||(a,b) - (a',b)|| = \sqrt{(a-a')^2 + (b-b)^2} = |a - a'| < \epsilon$. Thus $B_\epsilon((a,b)) \not\subset A$, and A can contain no ball, and hence has empty interior.

For similar reasons, $(A^c)^\circ = \emptyset$.

Now, for any set $E \subset \mathbb{R}^n$, we have $\overline{E} = ((E^c)^\circ)^c$ (checking this is a good way to make sure you understand the definitions of closure and interior; just show that the two sets are subsets of one another). From this and our above results, we conclude that $\overline{A} = \emptyset^c = \mathbb{R}^2$, $\overline{A^c} = \emptyset^c = \mathbb{R}^2$, and hence $\overline{A} \cap \overline{A^c} = \mathbb{R}^2$.

(b) $B^\circ = B$, since B is open.

$\overline{B} = \{x \in \mathbb{R}^2 | \; ||x|| \leq 1\}$. We can see this as follows. If $||x|| < 1$, then $x \in B \subset \overline{B}$. If $||x|| > 1$, then $||x|| = 1 + \delta$ where $\delta > 0$. We have $B_{\delta/2}(x) \subset B^c$, and thus $x \notin \overline{B}$. Finally, if $||x|| = 1$, then if we define $x_n = \frac{n-1}{n}x$ where $n \in \mathbb{N}$, then $x_n \in B$ (since $||x_n|| = \frac{n-1}{n} < 1$) and $x_n \to x$, and so $x \in \overline{B}$.

For similar reasons, we have $(B^c)^\circ = \{x \in \mathbb{R}^2 | \; ||x|| > 1\}$ and $\overline{B^c} = \{x \in \mathbb{R}^2 | \; ||x|| \geq 1\} = B^c$. We therefore have $\overline{B} \cap \overline{B^c} = \{x \in \mathbb{R}^2 | \; ||x|| = 1\}$, the unit circle.

(c) $(A \cap B)^\circ = \emptyset$, since $(A \cap B)^\circ \subset A^\circ = \emptyset$. Finally, for similar reasons as $\overline{A} = \mathbb{R}^2$, we have $\overline{(A \cap B)} = \overline{B}$