Solution Set 1E

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Math 23a
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12 (E) Let $\{S_n\}$ be a collection of open sets in $\mathbb{R}^n$, and let $\{T_n\}$ be a collection of closed sets. Show that:

(a) $S_1 \cap S_2$ is open.

Solution: Consider $x \in S_1 \cap S_2$. If $x \in S_1 \cap S_2$, then $x \in S_1$, and as S_1 is open, $\exists \epsilon_1 > 0$ s.t. $B_{\epsilon_1}(x) \subset S_1$. Likewise, $x \in S_2$, and as S_2 is open, $\exists \epsilon_2 > 0$ s.t. $B_{\epsilon_2}(x) \subset S_2$. Hence, denote $M = \min\{\epsilon_1, \epsilon_2\}$. Then certainly $B_M(x) \subset S_1$, and $B_M(x) \subset S_2$, so $B_M(x) \subset S_1 \cap S_2$. Hence, $S_1 \cap S_2$ is open.

(b) $\bigcup S_n$ is open (note that this symbol denotes the *arbitrary* (i.e. countably infinite) union of the S_n).

Solution: Take $x \in \bigcup S_n$. Then $x \in S_i$ for some $i \in \mathbb{N}$. But S_i is open, so $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \subset S_i$. Hence, $B_\epsilon(x) \subset \bigcup S$, so $\bigcup S_n$ is open.

(c) $T_1 \cup T_2$ is closed.

Solution: We note that $(T_1 \cup T_2)^c = (T_1)^c \cap (T_2)^c$, as $x \in (T_1 \cup T_2)^c \iff x \in (T_1)^c$ and $x \in (T_2)^c \iff x \in (T_1)^c \cap (T_2)^c$. Since $(T_1)^c \cap (T_2)^c$ is an intersection of open sets, by (a), $(T_1)^c \cap (T_2)^c$ is open, and so $(T_1 \cup T_2)^c$ is open, and hence $T_1 \cup T_2$ is closed.
(Please note that the $\iff$ symbol is shorthand for "if and only if" - invoking it saves time in working through both directions of a set equality.)

(d) $\bigcap T_n$ is closed (note that this symbol denotes the *arbitrary* intersection of the T_n).

Solution: Noting that $x \in (\bigcap T_n)^c \iff x \in (T_k)^c$ for some $k \in \mathbb{N} \iff x \in \bigcup (T_n)^c$, we have that $(\bigcap T_n)^c = \bigcup (T_n)^c$. Since $\bigcup (T_n)^c$ is an arbitrary union of open sets, we have from (b) that $\bigcup (T_n)^c$ is open, and hence $(\bigcap T_n)^c$ is open, so $\bigcap T_n$ is closed.

Notes: One of my goals for this semester is to shorten my notes considerably. Keeping comments to a minimum....

1. The "main" point of this problem, which was possibly confused by notation, was to show that the *arbitrary* union of open sets is open and that the *arbitrary* intersection of closed sets is closed. Notice that the arbitrary *intersection* of open sets need not be open, for example. Hence, induction was not a viable option for parts (b) or (d), as induction only deals with finite cases. Note that you can use induction to prove that any *finite* intersection of open sets is open, for example.

2. I apologize if the "if and only if" notation is confusing. I would say, however, that this is a good way to save time, particularly on exams.