

## 2A

**Claim (a):** The integers as a subset of the reals are discrete.

**Proof:** For any integer  $x$ , choose  $B_\epsilon(x)$  with  $\epsilon \leq \frac{1}{2}$ . Then such a ball will only intersect with the set of integers at the point  $x$ . ♡

**Claim (b):** Any function  $f : S \rightarrow \mathbb{R}$  where  $S$  is discrete is continuous.

**Proof:** Since  $S$  is discrete, there is an  $\epsilon$  such that for any  $x \in S$ ,  $B_\epsilon(x) \cap S = \{x\}$ . Thus choosing  $\delta = \epsilon$ , we have that for  $x$  and  $y$  in  $S$ ,  $|x - y| < \delta \Rightarrow y = x$ . Then  $|f(x) - f(y)| = |f(x) - f(x)| = 0$ , which satisfies the conditions for continuity vacuously. ♡

**Part C:** Give examples of sets that illustrate the necessity of the three conditions of boundedness, discreteness, and closedness for the Bolzano-Weirstrauss theorem to imply finiteness.

**Answer:**  $S = \{\frac{1}{n}\}$  where  $n$  is a natural number is bounded in the unit interval, discrete (for any  $x$  in this set, think of  $B_{x^3}$  or something small and dependent on  $x$ ), but it is infinite, since it is not closed—a limit point is zero and it is not in the set.

The natural numbers are discrete, closed, but not bounded, thus are infinite.

The unit interval is closed and bounded, but is not discrete, thus is infinite.

♡