SOLUTION SET 2B

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6. Let $A \subset \mathbb{R}^n$ be compact, and suppose that $B \subset A$ is closed. Use the "open cover" definition to show that B is compact.

Let $\{S_\alpha\}_{\alpha \in I}$ be an open cover of B. Since B is closed, B^c is open. The collection $\{S_\alpha\}_{\alpha \in I}$ together with B^c forms an open cover of A, which, being compact has a finite subcover. Now, this finite subcover will certainly cover B since $B \subset A$. It may or may not include B^c (which is not a priori a member of our original open cover); if it does not, we're done since we've found a finite subcollection of $\{S_\alpha\}_{\alpha \in I}$ that covers B. If it does, just take all the members of the finite subcollection except B^c; this new subcollection will also cover B since B and B^c are disjoint. In any case, given an open cover of B we have found a finite subcover, so B is compact.

Now, the fact that the problem did not say what B is open *in* may have been confusing (for example, $(0, 1/2]$ is closed in $(0, 1)$ but not in $\mathbb{R}$). But the point is it does not matter. For, as you can check, if B is closed in A, $B = C \cap A$ where C is closed in $\mathbb{R}^n$. And A, being compact, is closed in $\mathbb{R}^n$, so B is a finite intersection of closed subsets of $\mathbb{R}^n$, and is thus closed in $\mathbb{R}^n$.

In general, we say that compactness behaves well under taking of subspaces; if X is a metric space, and $Y \subset Z$ is a subspace, then a set $A \subset Y$ is compact in X iff it is compact in Y. Thus, compactness is something intrinsic, and does not depend on the ambient space. This is extremely important.

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