Math 23b Solution: Problem C

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We have $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous, and $A \subset \mathbb{R}^m$ is compact. We will show that $f(A)$ is compact. Let $\{S_\alpha\}_{\alpha \in I}$ be an open cover of $f(A)$ (possibly uncountable, if I, the index set, is uncountable). Then since f is continuous, the sets $\{f^{-1}(S_\alpha)\}_{\alpha \in I}$ are open. In fact, they form a cover for A, since if $x \in A$, then $f(x) \in S_\alpha$ for some $\alpha \in I$, and so $x \in f^{-1}(S_\alpha)$. Since A is compact, there exists a finite subcover $\{S_{\alpha_1}, \dots, S_{\alpha_N}\}$. Then $f(A) \subset S_{\alpha_1} \cup \dots \cup S_{\alpha_N}$. Indeed, let $y \in f(A)$. Then $y = f(x)$ for some $x \in A$, and so $x \in f^{-1}(S_{\alpha_i}$ for some $1 \leq i \leq N$. Then $y \in S_{\alpha_i}$. Thus every open cover of $f(A)$ has a finite subcover, and $f(A)$ is compact

The most common error was to assert that $f(f^{-1}(S_{\alpha_i}))$ was the requisitie subcover. This is only true if $f(f^{-1}((S_{\alpha_i})) = S_{\alpha_i}$, which is not in general true (take, for instance, f to be a constant map). What we do know is that $f(f^{-1}((S_{\alpha_i})) \subset S_{\alpha_i}$, which is true for any map f, and is all that we need.