SOLUTION SET 3B

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MATH23B
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6. Show that $O_n(\mathbb{R})$ is compact in $M_n(\mathbb{R})$ by showing that:

(a) $O_n(\mathbb{R})$ **is closed in** $M_n(\mathbb{R})$.

Consider the map $f : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ defined by $f(A) = A^tA$. We argue that f is continuous. The map $A \mapsto A$ is clearly continuous, being the identity. Also, g, the function that sends A to A^t is continuous. To see this, notice that given any matrix C, $\|C^t\| = \|C\|$, as C and C^t have the same entries, only rearranged in a particular way. Now, $B^t - A^t = (B - A)^t$, and so

$$\begin{aligned} \|g(B) - g(A)\| &= \|B^t - A^t\| && (1)\\ &= \|(B - A)^t\| && (2)\\ &= \|B - A\| && (3) \end{aligned}$$

So, whenever $\|B - A\| < \epsilon$, $\|g(B) - g(A)\| < \epsilon$.

Being the product of two continuous functions, f is continuous.

Now, by definition, $O_n(\mathbb{R}) = f^{-1}(I)$, and since I is closed in $M_n(\mathbb{R})$ and f is continuous, $O_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$.

A whole lot of people tried considering $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$, and arguing (incorrectly) that $O_n(\mathbb{R})$ is the preimage of $\{\pm 1\}$ under the determinant map. But $\det(A) = \pm 1$ does not imply that A is orthogonal, as the following matrix shows:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

This is a very important counterexample, so please remember it. The general principle to keep in mind from this is that if $f : X \longrightarrow Y$ is a map, and $B \subset X$, $f^{-1}(f(B)) \supset B$, but is not necessarily equal.

Similarly, a few considered (essentially) the determinant map with domain $O_n(\mathbb{R})$, in other words the restriction of the map to the orthogonal group. In this case, yes, the preimage of $\{\pm 1\}$ is $O_n(\mathbb{R})$ but this tells you only that $O_n(\mathbb{R})$ is closed *in itself*. Since we want it closed in $M_n(\mathbb{R})$ we need to use the larger space as our domain.

(b) $O_n(\mathbb{R})$ **is bounded in** $M_n(\mathbb{R})$.

Let $A \in O_n(\mathbb{R})$. We saw on the preview problems that the columns of an orthogonal matrix are orthonormal. In particular, for the columns to have length

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1 in $\mathbb{R}^n$, no entry of A can be more than 1 in absolute value. Then, considering the canonical bijection between $M_n(\mathbb{R})$ and $\mathbb{R}^{n^2}$, we see that $O_n(\mathbb{R})$ will be completely contained in the "box" $[-1,1]^{n^2}$ so it is bounded.
We can be more fancy by actually trying to compute $\|A\|$. We get

$$\|A\|^2 = \sum_{i=1}^{n} {a_{i,1}}^2 + \ldots + \sum_{i=1}^{n} {a_{i,n}}^2 \tag{4}$$

$$= \underbrace{1 + \ldots + 1}_{n} \tag{5}$$

$$= n \tag{6}$$

This works for any $A \in O_n(\mathbb{R})$. So, the norm of any matrix in $O_n(\mathbb{R})$ is $\sqrt{n}$, and hence $O_n(\mathbb{R})$ is bounded.

Being closed and bounded in $M_n(\mathbb{R})$, $O_n(\mathbb{R})$ is compact.