Math 23b Solution: Problem D

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We know that $det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous. We also know from first semester that A is invertible iff $det(A) \neq 0$, or in other words $GL_n(\mathbb{R}) = det^{-1}(\{0\}^c)$. Now, sets with only one member are clearly closed, since the only sequence of elements is the constant sequence. Thus $\{0\}$ is closed, and $\{0\}^c$ is open. Since the determinant is continuous, this means that $GL_n(\mathbb{R})$ is open.

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The set $\{1\} \subset \mathbb{R}$ is closed, and hence $SL_n(\mathbb{R}) = det^{-1}(\{1\})$ is closed (since the inverse image of closed sets is closed). To show that $SL_n(\mathbb{R})$ is unbounded for $n \geq 2$, let $M > 0$. We will construct an $A \in SL_n(\mathbb{R})$ with $||A|| > M$, which will show that $SL_n(\mathbb{R})$ is unbounded. Define A as follows: let $a_{11} = M$, (where of course a_{ij} is the $(i\ j)$-th component of A), $a_{22} = 1/M$, (here is where we need $n \geq 2$) $a_{ii} = 1$ for $i \geq 3$, and zeroes in all the off-diagonal elements. Then since A is diagonal, the determinant is the product of the diagonal elements, which is $M/M = 1$, so that $A \in SL_n(\mathbb{R})$. However,

$$||A|| = \sqrt{M^2 + (1/M)^2 + \cdots + 1} > \sqrt{M^2} = M.$$