Math 23b Solution: Problem 3E

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8

X is neither open nor closed. To show it is not open, first note that $I \in X$, since it is already diagonal. Define $I_n = \begin{bmatrix} 1 & 1/n \\ 0 & 1 \end{bmatrix}$. Each I_n is in Jordan block form, and since Jordan block form is unique these cannot be diagonalizable. We have $||I_n - I|| = 1/n$, and since $1/n \to 0$ as $n \to \infty$ we have $I_n \to I$. Thus for any $\epsilon > 0$, for sufficiently large n (specifically, $n > 1/\epsilon$) $I_n \in B_\epsilon(I)$, and so X cannot be open

To show it is not closed, consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and let $A_n = \begin{bmatrix} 1 & 1 \\ 0 & 1 - 1/n \end{bmatrix}$. By a similar argument to before we have $A_n \to A$ as $n \to \infty$. However, the characteristic polynomial of A_n had two distinct roots (namely, 1 and $1 - 1/n$), and so has two linearly independent eigenvectors, which form a basis with respect to which A_n is diagonal. Thus A is a limit point of X not contained in X, so X is not closed either.