SOLUTION SET 4B

ALEX LEVIN
MATH 23B
PROF. BOLLER

3. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R} \longrightarrow \mathbb{R}^n$ be two differentiable curves with $f'(t) \neq 0$ and $g'(t) \neq 0$ for all $t \in \mathbb{R}$. Suppose that $p = f(s_0$ and $q = g(t_0)$ are closer than any other pair of points on the two curves. Prove that the vector $p - q$ is orthogonal to both velocity vectors $f'(s_0)$ and $g'(t_0)$.

Consider $\rho : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $\rho(s,t) = \|f(s) - g(t)\|^2$. Then, (s_0, t_0) must be a critical point of the function ρ, i.e. a point where both partial derivatives are 0 or do not exist. Now, $\rho(s,t) = (f(s) - g(t)) \cdot (f(s) - g(t))$, and so using the product rule to differentiate with respect to s and t we get:respect to s and t we get:

$$\frac{\partial \rho}{\partial s} = 2(f(s) - g(t)) \cdot f'(s) \tag{1}$$

$$\frac{\partial \rho}{\partial t} = -2(f(s) - g(t)) \cdot g'(t) \tag{2}$$

Throughout, the $\cdot$ represents the standard dot product Now, both curves are differentiable, so the above partial derivatives always exist, and so for (s_0, t_0) to be a critical point, $(f(s_0) - g(t_0)) \cdot f'(s_0) = 0$, and $(f(s_0) - g(t_0)) \cdot g'(t_0) = 0$, in other words $p - q$ is orthogonal to both $f'(s_0)$ and $g'(t_0)$, as desired.

To be perfectly careful you may want to check that if the curves actually intersect (i.e. the minimum distance is actually 0), then certainly $p - q = 0$ is orthogonal to both velocities.

(b) Apply the result in part (a) to find the closest pair of points for the "skew" straight lines in $\mathbb{R}^3$ defined by $f(s) = (s, 2s, -s)$ and $g(t) = (t+1, t-2, 2t+3)$.

Here, if $p = f(s_0)$, and $q = g(t_0)$, we set $f(s_0) - g(t_0) = (s_0 - (t_0 + 1), 2s_0 - (t_0 - 2), -s - (2t_0 + 3))$ orthogonal to $f'(s_0) = (1, 2, -1)$ and $g'(t_0) = (1, 1, 2)$ and get two equations in two unknowns, which we solve for s_0 and t_0. We obtain $s_0 = -41/35$, $t_0 = -36/35$. The closest points are thus $p = f(s_0) = (-41/35, -82/35, 41/35)$ and $q = g(t_0) = (-1/35, -106/35, 3/35)$.

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