SOLUTION SET 5B

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MATH 23B
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(1) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $f(x, y) = x^2 \sin(1/x) + y^2$ for $x \neq 0$ and $f(0, y) = y^2$.

(a) Show that f is continuous at $(0, 0)$.
Notice that given $\epsilon > 0$, for (x, y) with $x^2 + y^2 < \epsilon$, $|f(x, y)| < \epsilon$. Indeed, $|f(x, y)| = |x^2 \sin(1/x) + y^2| \leq |x^2 \sin(1/x)| + |y^2|$ for $x \neq 0$, and since the $\sin(1/x)$ can only be between -1 and 1, this is less than or equal to $|x|^2 + |y|^2 = \epsilon$. When $x = 0$, a similar argument applies only we do not have to worry about the pesky $\sin(1/x)$ term. In any case, we can choose $\delta = \sqrt{\epsilon}$ so that whenever $\|(x, y)\| < \delta$, $|f(x, y)| < \epsilon$, and as $f(0, 0) = 0$, f is continuous at 0.

(b) Find the partial derivatives of f at $(0, 0)$. We have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} \tag{1}$$

$$= \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} \tag{2}$$

$$= 0 \tag{3}$$

and also,

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y} \tag{4}$$

$$= \lim_{y \to 0} \frac{y^2}{y} \tag{5}$$

$$= 0 \tag{6}$$

Notice that we had to take a careful limit "bad behavior" of the function.

(c) Show that f is differentiable at $(0, 0)$.
Our computation of the partials suggests that the 0 map is a candidate for the differential of f at $(0, 0)$. Indeed,

$$\lim_{\|(x,y)\| \to 0} \frac{|f(x,y) - f(0,0) - 0|}{\|(x,y)\|} = \lim_{\sqrt{x^2+y^2} \to 0} \frac{|f(x,y)|}{\sqrt{x^2+y^2}} \tag{7}$$

$$\leq \lim_{\sqrt{x^2+y^2} \to 0} \frac{x^2+y^2}{\sqrt{x^2+y^2}} \tag{8}$$

$$= 0 \tag{9}$$

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Notice that in (**??**), I put absolute values around the numerator, which deviates slightly from the definition of differentiability that we gave in class. However, the result is equivalent, and our function is indeed differentiable at 0, with differential being the identically 0 function.

(d) Show that $D_1 f$ is not continuous at $(0, 0)$.
For (x, y) with $x \neq 0$, we have $D_1 f(x, y) = \frac{\partial f}{\partial x}(x, y) = 2x \sin(1/x) - \cos(1/x)$. Let us approach $(0, 0)$ by a path along the x axis (this involves sending x values to 0, and keeping y fixed at 0). Notice that $\lim_{x \to 0} (2x \sin(1/x) - \cos(1/x))$ does not even exist, since $2x \sin(1/x) \to 0$ (after all, $\sin(1/x)$ is bounded), while x is getting arbitrarily small, though $\cos(1/x)$ is bouncing around between 1 and -1. Explicitly, at $x = 1/(x\pi)$, $\cos(1/x) = (-1)^n$. Thus, since we've found a particular path along which $\frac{\partial f}{\partial x}(x, y)$ does not have a limit as $(x, y) \to (0, 0)$, $D_1 f$ cannot be continuous at $(0, 0)$.