Math 23b Solution: Problem D

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I claim that f is differentiable at the origin, with derivative the zero map $L = 0$. Indeed, we have

$$\lim_{||h||\to 0} \frac{f(0+h) - f(0) - L(h)}{||h||} = \lim_{||h||\to 0} \frac{h||h||}{||h||} = \lim_{||h||\to 0} h = 0$$

Since the total derivative is equal to the zero map, all of the partial derivatives must also be zero at the origin.

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(a) Let p/q be rational, and let x_i be a sequence of irrational number converging to p/q (one example would be $x_n = p/q + \pi/n$). Then $f(x_i) = 0$ for all i and so $f(x_i) \to 0$ but $f(p/q) = 1/q^3 \neq 0$. Thus f is not continuous at any rational.
(b) This part was hard. We begin with a classic result
Lemma: Let $f(x) \in \mathbb{Z}[x]$ (i.e. a polynomial with integer coefficients) have degree n and no rational roots, and let $\alpha \in \mathbb{R}$, $f(\alpha) = 0$. Then there exists $C > 0$ such that for any rational p/q with $q > 0$ we have $|\alpha - p/q| > C/q^n$.

Intuitively, what this theorem asserts is that real numbers that are roots of polynomials with integer coefficents cannot be approximated "too quickly" by rationals.
Proof: Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_i \in \mathbb{Z}$. Then for any rational p/q with $q > 0$, we have

$$|f\left(\frac{p}{q}\right)| = |a_n \left(\frac{p}{q}\right)^n + \cdots + a_1 \left(\frac{p}{q}\right) + a_0| = \left|\frac{1}{q^n}\right| \cdot |a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 p^{n-1} q + a_0 q^n|$$

The first term in the equation on the right is simply $1/q^n$ since $q > 0$, while the second term is an integer (since p, q and a_i are all integers), and is non-zero (since if it were zero $f(p/q) = 0$) so must be greater than or equal to one. Hence $|f(p/q)| \geq 1/q^n$.

Now, let $\delta > 0$. The function $|f'(x)|$ is a continuous function, and hence there exists $M > 0$ such that $|f'(x)| < M$ for all x in the compact set $[p/q - \delta, p/q + \delta]$. We now examine two cases for p/q.

Case 1: $|\alpha - p/q| > \delta$. In this case we have $|\alpha - p/q| > \delta/q^n$ since $q > 1$.
Case 2: $|\alpha - p/q| \leq \delta$. By the mean value theorem there is a β between α and p/q such that $f(p/q) = f(p/q) - f(\alpha) = f'(\beta)(p/q - \alpha)$. Note that $f'(\beta) \neq 0$ since $f(p/q) \neq 0$ and $|f'(\beta)| < M$. Thus

$$|\alpha - \frac{p}{q}| = \frac{|f\left(\frac{p}{q}\right)|}{|f'(\beta)|} > \frac{1}{Mq^n}$$

So letting $C = \min(1/M, \delta)$ we have $|\alpha - p/q| > C/q^n$ in either case. □

Now onto the main problem. Note that $\sqrt{2}$ is a root of $x^2 - 2$, and so the previous lemma applies for some $C > 0$ (in fact, we can take $C = 1$ in this case). I claim that $F_3'(\sqrt{2}) = 0$. Since $f_3(\sqrt{2}) = 0$, this amounts to showing that $\lim_{h \to 0} f_3(\sqrt{2} + h)/h = 0$. We consider two cases
Case 1: $\sqrt{2} + h$ is irrational. In this case $f_3(\sqrt{2} + h) = 0$ and we are done.
Case 2: $\sqrt{2} + h = p/q \in \mathbb{Q}$ In this case by the lemma $|h| = |\sqrt{2} - p/q| > C/q^2$. Also, by definition $f_3(\sqrt{2} + h) = 1/q^3$. hence

$$\left|\frac{f_3(\sqrt{2} + h)}{h}\right| \leq \frac{1/q^3}{C/q^2} = \frac{1}{Cq}$$

As $h \to 0$, $q \to \infty$ (this was part of problem 5 on the first problem set of 23b; see the solution on-line for the proof of this fact). Thus as $h \to 0$, the quotient approaches 0 whether $\sqrt{2} + h$ is rational or irrational. Hence $f_3'(\sqrt{2}) = 0$.