Solution Set 5E

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Math 23a
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10 (E) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $f(x,y) = x \cdot y \cdot \frac{(x^2-y^2)}{(x^2+y^2)}$ for $(x,y) \neq (0,0)$, and $f(0,0) = 0$.

(b) Show that $D_1 f(0,y) = -y$ and $D_2 f(x,0) = x$ for all $x, y \in \mathbb{R}$.

Solution: Fix $y \in \mathbb{R}$. Then

$$D_1 f(0,y) = \lim_{h \to 0} \frac{f((0,y) + h(1,0)) - f(0,y)}{h}$$

$$= \lim_{h \to 0} \frac{f((h,y)) - f(0,y)}{h}$$

$$= \lim_{h \to 0} \frac{h \cdot y \cdot \frac{h^2-y^2}{h^2+y^2} - 0 \cdot y \cdot \frac{0^2-y^2}{0^2+y^2}}{h}$$

$$= \lim_{h \to 0} y \cdot \frac{h^2 - y^2}{h^2 + y^2} = -y.$$

Likewise,

$$D_2 f(x,0) = \lim_{h \to 0} \frac{f((x,0) + h(0,1)) - f(x,0)}{h}$$

$$= \lim_{h \to 0} \frac{f((x,h)) - f(x,0)}{h}$$

$$= \lim_{h \to 0} \frac{x \cdot h \cdot \frac{x^2-h^2}{x^2+h^2} - x \cdot 0 \cdot \frac{x^2-o^2}{x^2+0^2}}{h}$$

$$= \lim_{h \to 0} x \cdot \frac{x^2 - h^2}{x^2 + h^2} = x$$

(c) Show that $D_2D_1f(0,0)$ and $D_1D_2f(0,0)$ exist but are not equal.

Solution: We again proceed directly through the limit definition. Note that

$$\begin{aligned} D_2D_1f(0,0) &= \lim_{h\to 0} \frac{D_1f((0,0)+h(0,1)) - D_1f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{D_1f(0,h) - D_1f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{-h-0}{h} = -1. \end{aligned}$$

Likewise,

$$\begin{aligned} D_1D_2f(0,0) &= \lim_{h\to 0} \frac{D_2f((0,0)+h(1,0)) - D_2f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{D_2f(h,0) - D_2f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h-0}{h} = 1. \end{aligned}$$

Hence, $D_2D_1f(0,0) = -1$ and $D_1D_2f(0,0) = 1$, so $D_2D_1f(0,0)$ and $D_1D_2f(0,0)$ exist but are not equal.

Notes: No real problems here - scores were very high. The take away message is that if a function is not C^2 (formally, we say that $f \in C^n$ if its n-th order partials exist and are continuous), the cross partials need not be equal.