6A SOLUTIONS

a) Here is a simple condition that makes $f(x,y) = ax^2+bxy+cy^2+dx+ey+$ harmonic: $a=-c$, since taking $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ will annihilate any terms in x, y with degree <2, leaving $2a+2c=0$.

b.) Note that $f(\vec{x}) = \|\vec{x}\|^{2-n}$

$$\frac{\partial f}{\partial x_i} = \frac{2-n}{2}\left(x_1^2+\dots+x_n^2\right)^{-n/2}\cdot 2x_i = (2-n)(x_i)\left(\sum x_i^2\right)^{-n/2}$$

$$\frac{\partial^2 f}{\partial x_i^2} = (2-n)\left(-\tfrac{n}{2}\right)(2x_i)(x_i)\left(\sum x_i^2\right)^{\frac{-n-2}{2}} + 2-n\left(\sum x_i^2\right)^{-n/2}$$

$$\nabla^2 f = \sum_i \frac{\partial^2 f}{\partial x_i^2} = \left[(2-n)\left(\sum x_i^2\right)^{-n/2}\right]\left[\left(\sum x_i^2\right)^{-1}\left(\sum x_i^2\right)(-n)+\sum 1\right]$$

$= 0$ since $-n+n=0$ here

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c.) let $h(x,y) = (e^x\cos y, e^x\sin y)$.

Then $f = g\circ h$, and hence, by the chain rule,

$$Jf = Jg\cdot Jh = \begin{bmatrix}\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}\begin{bmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y\end{bmatrix}$$

$$= \left(\frac{\partial g}{\partial x}e^x\cos y + \frac{\partial g}{\partial y}e^x\sin y,\ -\frac{\partial g}{\partial x}e^x\sin y + \frac{\partial g}{\partial y}e^x\cos y\right)$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = e^x\left(\frac{\partial g}{\partial x}\cos y\right) + e^x\left(\frac{\partial^2 g}{\partial x^2}\cos y\right) + e^x\left(\frac{\partial g}{\partial y}\sin y\right) + e^x\left(\frac{\partial^2 g}{\partial x\partial y}\sin y\right)$$

By the product rule. Similarly,

$$\frac{\partial^2 f}{\partial y^2} = -e^x\left(\frac{\partial g}{\partial x}\cos y\right) - e^x\left(\frac{\partial^2 g}{\partial x\partial y}\sin y\right) - e^x\left(\frac{\partial g}{\partial y}\sin y\right) + e^x\left(\frac{\partial^2 g}{\partial y^2}\cos y\right)$$

h is $C^\infty \Rightarrow \frac{\partial^2 g}{\partial x\partial y} = \frac{\partial^2 g}{\partial y\partial x}$, so terms cancel with cross partials, leaving

$$\nabla^2 f = e^x\cos y\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}\right) = e^x\cos y\,(\overset{\text{by hypothesis}}{0}) = 0$$

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