Math 23b Solution: Problem D

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March 18, 2005

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Let $f_i(\mathbf{x})$ be the i'th component of $f(\mathbf{x}) = ||\mathbf{x}||\mathbf{x}$, so that $f_i(\mathbf{x}) = x_i||\mathbf{x}||$. A simple computation shows that

$$\frac{\partial f_i}{\partial x_i}(\mathbf{x}) = ||\mathbf{x}|| + \frac{x_i^2}{||\mathbf{x}||}$$

while if $j \neq i$ then

$$\frac{\partial f_i}{\partial x_j}(\mathbf{x}) = \frac{x_i x_j}{||\mathbf{x}||}.$$

These are all for $\mathbf{x} \neq \mathbf{0}$; for $\mathbf{x} = \mathbf{0}$, we showed in Part D of the previous problem set that all the partial derivatives are equal to 0.

We now look at the second partials at the origin. We first look at $\frac{\partial f_i}{\partial x_i}(\mathbf{x})$ in a direction e_j with $j \neq i$. The putative second partial at the origin is

$$\lim_{h\to 0} \frac{\frac{\partial f_i}{\partial x_i}(he_j) - \frac{\partial f_i}{\partial x_i}(\mathbf{0})}{h} = \lim_{h\to 0} \frac{||he_j|| + 0^2/||he_j||}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

and this limit does not exist, since it approaches 1 from the right and -1 from the left. Thus these partials do not exist. Now we consider the partial of $\frac{\partial f_i}{\partial x_i}(\mathbf{x})$ in the e_i direction:

$$\lim_{h\to 0} \frac{\frac{\partial f_i}{\partial x_i}(he_i) - \frac{\partial f_i}{\partial x_i}(\mathbf{0})}{h} = \lim_{h\to 0} \frac{||he_i|| + h^2/||he_j||}{h} = \lim_{h\to 0} \frac{|h|}{h} + \frac{h}{|h|}$$

and this also does not exist, since it approaches 2 from the right and -2 from the left. So none of the partials of $\frac{\partial f_i}{\partial x_i}(\mathbf{x})$ exist. We now consider $\frac{\partial f_i}{\partial x_j}(\mathbf{x})$ in any direction e_k:

$$\lim_{h\to 0} \frac{\frac{\partial f_i}{\partial x_j}(he_k) - \frac{\partial f_i}{\partial x_i}(\mathbf{0})}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

The numerator in the middle equation is zero because at most one of x_i and x_j will be non-zero, and hence $x_i x_j = 0$. Hence all of these second partials do exist, and are equal to 0. So some of the partials do exist, some do not.