SOLUTION SET 7B

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MATH 23B
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(6) **In the following problem, we consider the notion of the** *local invertability* **of a function and the relationship between this condition and that of injectivity.**

(a) **Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is locally invertible at every point of its domain (that is, $f'(a) \neq 0, \forall a \in \mathbb{R}$). Show that f is one-to-one.**
Suppose $f(x) = f(y)$ for $x, y \in \mathbb{R}$, with $x < y$. Then by Rolle's Theorem (or the Mean Value Theorem) there exists $c \in (x, y)$ such that $f'(c) = 0$, which contradicts local invertibility. Thus f is one-to-one.
A relatively common mistake was saying that because f' had to exist, but was nowhere 0, f' had to be either positive or negative everywhere, and thus the function was either monotonically increasing or decreasing. This is true, however the proof is not immediate. I imagine people thought about an Intermediate Value Theorem argument for f', but f' need not be continuous given the problem statement, so the theorem does not apply.

(b) **Consider $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by $g(x, y) = (e^x \cos y, e^x \sin y)$. Show that g is locally invertible at every point of its domain (that is, $\det[Jg(\boldsymbol{x})] \neq 0, \forall \boldsymbol{x} \in \mathbb{R}^2$), but that g is not one-to-one**
We have

$$Jg(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

and

$$\det Jg(x, y) = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x}$$

which is not equal to 0 for any x. Hence, g is locally invertible.
However, $g(x, y + 2\pi) = g(x, y)$ for all x and y, so g is not one-to-one.

(7) **Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $f(x, y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$. Show that f is locally invertible in a neighborhood of every point except the origin and compute f^{-1} explicitly.**

Our function is continuously differentiable on $\mathbb{R} \setminus \{(0, 0)\}$ (it is not defined at the origin). A quick computation shows that for $(x, y) \neq (0, 0)$,

$$Jf(x, y) = \begin{pmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-2xy}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{pmatrix}$$

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and $\det Jf(x,y) = -1/(x^2+y^2)^2$ which is strictly negative (and hence nonzero) for $(x,y) \neq (0,0)$. Using the Inverse Function Theorem we establish that f is locally invertible in a neighborhood of every point except the origin.

Now we compute f^{-1}. Let $f^{-1}(x,y) = (u,v)$. Then $ff^{-1}(x,y) = (x,y)$ implies that $u/(u^2+v^2) = x$, and $v/(u^2+v^2) = y$. Hence, $x^2+y^2 = 1/(u^2+v^2)$ and substituting in for $1/(u^2+v^2)$, we see that $(x^2+y^2)v = y$, hence $v = y/(x^2+y^2)$, and similarly that $(x^2+y^2)u = x$, so that $u = x/(x^2+y^2)$. Hence $f^{-1}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$. In other words, f is its own inverse.

Perhaps an easier way to do this is to notice that f satisfies $f(\boldsymbol{x}) = \boldsymbol{x}/\|\boldsymbol{x}\|^2$ for $\boldsymbol{x} \in \mathbb{R}^2$ and to take it from there.