Math 23b Solution: Problem D

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(a) Define $h : \mathbb{R}^2 \to \mathbb{R}$ by $h(x, y) = 1+xy+x^2$, so that $F(\mathbf{x}) = (\varphi(\mathbf{x}), \psi(\mathbf{x}), h(\varphi(\mathbf{x}), \psi(\mathbf{x})))$. So that I don't spend the next 5 years typing this up I will suppress the $\mathbf{x}$ and I will use subscripts to denote partial derivatives, i.e. $\psi_2 = \frac{\partial \psi}{\partial x_i}(\mathbf{x})$. At any point the Jacobian of F is, using the chain rule to compute the bottom row,

$$\begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \psi_1 & \psi_2 & \psi_3 \\ h_1\varphi_1 + h_2\psi_1 & h_1\varphi_2 + h_2\psi_2 & h_1\varphi_3 + h_2\psi_3 \end{bmatrix}$$

Note that the last row is h_1 times the first row plus h_2 times the second row. Thus at any point the rows are linearly dependent, and so the determinant is equal to 0. Thus the main assumption of the inverse function theorem–the invertibility of the Jacobian–is not satisfied anywhere.

(b) We have shown that the assumptions of the inverse function theorem do not hold; now we will show explicitly that the conclusion does not either. One of the conclusions of the inverse function theorem was that if U was a sufficiently small open ball around a point $\mathbf{x}$, then $f(U)$ is an open ball around $f(\mathbf{x})$. We will show, on the contrary, that im F can contain no open balls. Let $\mathbf{x} \in \text{im } F$, so that $\mathbf{x} = (x, y, z) = (x, y, h(x, y))$. Note that for a given $(x, y) \in \mathbb{R}^2$ there is at most one z such that $(x, y, z) \in \text{im } F$ since we must have $z = h(x, y)$. Thus for any $\epsilon > 0$, the point $(x, y, z + \epsilon/2) \in B_\epsilon(\mathbf{x})$ but it is not conained in im F. Thus F can contain no open balls, and the conclusion of the inverse function theorem does not hold.

Most people had the right idea for this problem (although some people did not have very pretty part a)'s). One common solution to part b) was to assert that the image of F was a 2-manifold, and so could not contain any open balls in $\mathbb{R}^3$. There were a few problems with this: we had not yet defined a manifold when people did this problem; the image does not actually have to be a 2-manifold, depending on the specific nature of the functions; and most interestingly, we haven't proved that a 2-manifold cannot also be a 3-manifold! It is of course true that a manifold has a unique dimension, but proving this, in the most general setting, is not a triviality. It is not too difficult to prove the uniqueness of dimension for differentiable manifolds, but for merely continuous manifolds one needs a fair amount of algebraic topology. In any case, the proof for the specific case given above is much simpler.