Math 23b Solution: Problem D

David Jackson-Hanen

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a is a critical point of f, so that $\nabla f(a) = 0$. Hence by Taylor's theorem we have

$$f(a+h) - f(a) = q(h) + R_2(h)$$

where q is a quadratic form. Now consider q restricted to S^{n-1}, the unit sphere in $\mathbb{R}^n$. q is then a continuous function on a compact set, and so attains some mimimum value c on S^{n-1} (in fact, we showed in class that c was actually the smallest eigenvalue of the matrix associated to q). Since q is positive definite, we have $c > 0$.

By Taylor's theorem,

$$\lim_{||h|| \to 0} \frac{R_2(h)}{||h||^2} = 0$$

In particular, there is a $\delta > 0$ such that if $0 < ||h|| < \delta$, then $\left| \frac{R_2(h)}{||h||^2} \right| < \frac{c}{2}$. Using the the fact that q is a quadratic form and that $h/||h|| \in S^{n-1}$, we have

$$\frac{q(h) + R_2(h)}{||h||^2} = q\left(\frac{h}{||h||}\right) + \frac{R_2(h)}{||h||^2} > c - \frac{c}{2} > 0$$

for $0 < ||h|| < \delta$. Thus $q(h) + R_2(h) >$, and so $f(a+h) > f(a)$. Thus a is a local min as desired.

Most of the solutions were very good. A few people made some minor errors– for instance, some people tried to prove the theorem only using the fact that $R_2(h) \to 0$ as $h \to 0$, while you really have to use the fact that it goes to zero faster than $||h||^2$. But most people had the right idea.