SOLUTION SET 9B

ALEX LEVIN
MATH 23B
PROF. BOLLER

3. In class we proved the following theorem

> **Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \to \mathbb{R}$ be a bounded function. If f is continuous at a, then $o(f, a) = 0$.**

Prove the converse.

If we have $o(f, a) = 0$, then by definition $\lim_{\delta \to 0} (M(a, f, \delta) - m(a, f, \delta)) = 0$. Then, certainly $|f(x) - f(a)| < M(a, f, \delta) - m(a, f, \delta)$ for x within δ of a. And, because given $\epsilon > 0$, we can find a δ such that $|f(x) - f(a)| < M(a, f, \delta) - m(a, f, \delta) < \epsilon$ whenever $x - a < \delta$, continuity follows.

4. Let $A \subset \mathbb{R}^n$ be a closed rectangle and let $f : A \to \mathbb{R}$ be a bounded function. If P is a partition of A and P' is a refinement of P show that $L(f, P) \leq L(f, P')$

Let us consider one subrectangle of P, call it S. When we pass to P', this rectangle gets broken up into rectangles $S_1, \ldots, S_n$, such that $v(S_1) + \ldots + v(S_n) = v(S)$. But, for each i, $m_{S_i}(f) \geq m_S(f)$ by definition of $m_{S_i}(f)$ and $m_S(f)$ noting that each S_i is a subset of S. Summing over the S_i's gives us $m_s(f)v(S) = m_S(f)v(S_1) + \ldots + m_S(f)v(S_n)$ which is less than or equal to $m_{S_1}v(S_1) + \ldots + m_{S_n}v(S_n)$.

Considering in turn all $S \in P$ and performing a similar argument, and summing the results, gives us what we want to prove.

Date: April 22, 2005.