Math 23b Solution: Problem D

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(a) We will show that for any $a \in A$, $\lim_{z\to a} f(z) = 0$. This implies, by a simple calculation, that $o(f,a) = f(a)$ for any $a \in A$. To prove this limit, let $a \in A$, $a = (x,y)$. Fix $\epsilon > 0$, and choose $\delta > 0$ such that, if $|y - \frac{p}{q}| < \delta$ for $p,q \in \mathbb{Z}$ and $\frac{p}{q}$ in lowest terms, then $\frac{1}{q} < \epsilon$. We have proved several times on the homework that we can always find such a δ; see, for instance, the solution set to problem B on problem set 1 of math 23b. Then if $z \in B_\delta(a)$, by construction we either have $f(z) = 0$ or $f(z) = \frac{1}{N}$ with $\frac{1}{N} < \epsilon$; in either case, $f(z) < \epsilon$. This prove that $\lim_{z\to a} f(z) = 0$.
(b) We showed aove that $o(f,a) = f(a)$. This implies that f is continuous at a iff $f(a) = 0$. Now, f is non-zero precisely on the set, $\mathbb{Q}^2 \cap A$; since this set is countable, it has measure 0. Thus f is continuous except for a set of measure 0, and so must be integrable. Note that any square contained in A must contain at least one point on which $f = 0$; this implies that all the lower sums of f are equal to 0. Since the integral is the limit of the lower sums, this implies that the integral is equal to 0.