MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Stokes' Theorem

Stokes' Theorem.

Let S be a parametrized surface in $\mathbb{R}^3$. That is, suppose there exists some $D \subset \mathbb{R}^2$ that is open, connected, and simply-connected, with a boundary $C = \partial D$ that can be parametrized as a piece-wise smooth, positively-oriented curve, and a bijective function $\varphi : \overline{D} \to S$ that is continuous on $\overline{D}$ and continuously differentiable (except possibly on a set of measure zero) on D.

Let $F : U \to \mathbb{R}^3$ be a vector field that is class C^1 and represented by the components $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$, where $U \subset \mathbb{R}^3$ is some open set containing S.

Then Stokes' Theorem says that:

$$\int\int_S \text{curl } F = \int_C F$$

where, if

$$G = \text{curl } F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right),$$

then with the parametrization above,

$$\begin{aligned} \int\int_S \text{curl } F &= \int\int_S G \cdot \mathbf{n} \, dS \\ &= \int\int_D G(\varphi(u, v)) \cdot (\varphi_u \times \varphi_v) \, du \, dv, \end{aligned}$$

where $\mathbf{n}$ is the normal vector to the surface S, and u and v are the two variables parametrizing D as a subset of $\mathbb{R}^2$. (For more, please read Chapter 19.3 in Fitzpatrick or Chapter V, Section 6 of Edwards.)