Math 23b: Theoretical Linear Algebra and Multivariable Calculus II

MIDTERM EXAM 1- SOLUTIONS

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Problem	Points	Score
1	20	20
2	20	20
3	20	20
4	20	20
5	20	20
Total	100	100

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Please show all your work on this exam paper. You must show your work and clearly indicate your line of reasoning in order to get full credit. If you have work on the back of a page, indicate that on the exam cover.

Problem 1

Decide whether the following statements are True or False. (Note: There is no need to justify your answers. You get +5 for every correct answer and -2 for every wrong answer.)

T or F: If A is a non-empty and bounded subset of $\mathbb{R}$, then A has a maximal element.

– **False:** $(0, 1)$ is non-empty and bounded in $\mathbb{R}$, but it has no maximal element.

T or F: If the set A is a proper subset of B, then they don't have the same cardinality.

– **False:** $2\mathbb{N}$ is a proper subset of $\mathbb{N}$, but they have the same cardinality.

T or F: An open set cannot be closed.

– **False:** X and $\emptyset$ are both open and closed.

T or F: Let X, Y be metric spaces, let A be a closed subset of X and let $f : X \to Y$ be a continuous function. Then $f^{-1}(A)$ is closed.

– **True:** This is one of the equivalent definitions of continuous functions.

Problem 2

Prove or disprove the following statements:

(a) If a set S is a countable subset of a metric space, then the set of its limit points will be at most countable.

(b) Let $f : X \to \mathbb{R}$ be a continuous function between metric spaces. Let $Z(f) \subset X$ be the "zero-set" for f, namely the set of all x's in X such that $f(x) = 0$. Then $Z(f)$ is a closed set.

Solution.

(a) **False.** $\mathbb{Q}$ is a countable subset of $\mathbb{R}$, but its set of limit points is $\mathbb{R}$, which is not countable.

(b) **True.** Clearly $\{0\}$ is a closed subset of $\mathbb{R}$ and, by assumption, f is continuous. Hence $Z(f) = f^{-1}(\{0\})$ is also closed (by definition of continuous functions.

Problem 3

Given a metric space X and subset $A \subset X$, we define its *interior* A° as the set of all its interior points:

$$A^\circ \;=\; \{x \in X \,|\, x \text{ is interior point of } A\}$$

We say that $a \in X$ is a *boundary point* if $\forall \epsilon > 0$ we have both $S(a,\epsilon) \cap A \neq \emptyset$ and $S(a,\epsilon) \cap A^c \neq \emptyset$. The *boundary* of A is then

$$\partial A \;=\; \{x \in X \,|\, x \text{ is a boundary point of } A\}$$

Prove that:

(a) $A^\circ = A$ if and only if A is open,
(b) if $U \subset A$ and U is open, then $U \subset A^\circ$.
(c) $\partial A = \overline{A} - A^\circ$.

Solution.

(a) By definition, A is open if an only if every element of A is an interior point, which is the same as to say that $A = A^\circ$.
(b) Suppose $a \in U$. Since U is open, a is an interior point of U, hence there is $\epsilon > 0$ such that $S(a,\epsilon) \subset U \subset A$, which implies that a is an interior point of A, namely $a \in A^\circ$.
(c) By definition

$$\begin{aligned}
\partial A &= \left\{ a \in X \,\middle|\, \forall \epsilon > 0\,,\; S(a,\epsilon) \cap A \neq \emptyset \;\;\&\;\; S(a,\epsilon) \cap A^c \neq \emptyset \right\} \\
&= \left\{ a \in X \,\middle|\, a \text{ is a contact pt of } A \;\;\&\;\; a \text{ is not an interior pt of } A \right\} \\
&= \left\{ a \in X \,\middle|\, a\overline{A} \;\;\&\;\; a \notin A^\circ \right\} \;=\; \overline{A} - A^\circ
\end{aligned}$$

Problem 4

Let A be an $m \times n$ matrix and let b be a column m-vector. Consider the following function $f : \mathbb{R}^n \to \mathbb{R}^m$,

$$f(x) \; = \; Ax + b \; .$$

Prove that the derivative of f exists at every point $a \in \mathbb{R}^n$. What is $f'(a)$?

Solution.

The derivative of f is $f'(a) = A$ for every $a \in \mathbb{R}^n$. Indeed

$$f(a+h) - f(a) - Ah \; = \; A(a+h) - Aa - Ah \; = \; 0 \; ,$$

hence

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Ah}{||h||} \; = \; 0 \; .$$

Problem 5

We know that every finite subset S of a metric space X is automatically compact. What about countable sets? Prove or disprove the following statements:

(a) If $S \subset X$ is countable, then S is compact.
(b) If $S \subset X$ is countable, then S is not compact.

Solution.

(a) **False.** $\mathbb{Q}$ is a countable subset of $\mathbb{R}$ but it is not compact.
(b) **False.** The set $\{1/n \mid n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ is countable and compact.