Math 23b: Theoretical Linear Algebra and Multivariable Calculus II

MIDTERM EXAM 2 - solutions

April 17, 2006

Your name: ____________________

Problem	Points	Score
1	21	
2	20	
3	20	
4	20	
5	20	
Total	101	

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Please show all your work on this exam paper. You must show your work and clearly indicate your line of reasoning in order to get full credit. If you have work on the back of a page, indicate that on the exam cover.

Problem 1

Decide whether the following statements are True or False. (Note: There is no need to justify your answers. You get +3 for every correct answer and -1 for every wrong answer.)

T or F: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x = a$, then all its directional derivatives are continuous at $x = a$.

False

T or F: If $f : \mathbb{R}^2 \to \mathbb{R}^m$ is differentiable at (a, b), then $\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b)$.

False

T or F: A smooth function defined in the closed unit disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$ has a maximum in this disk.

True

T or F: A smooth function is equal to its Taylor series.

False

T or F: Let $R \subset \mathbb{R}^n$ be a closed rectangle. If $f : R \to \mathbb{R}$ is integrable on R, then f is continuous in the interior of R.

False

T or F: Every finite set $S \subset \mathbb{R}^n$ is Riemann-measurable.

True

T or F: If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then f is integrable on every bounded Riemann-measurable set $A \subset \mathbb{R}^n$.

True

Problem 2

Prove or disprove each of the following statements. In order to prove a statement, just provide a brief justification, while in order to disprove, you need to present a counterexample (no explanation is necessary, as long as the example is correct).

(a) Suppose the function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(1,1)$ and $D_h f(1,1) = D_k f(1,1) = 0$, where $h = (1,1)$ and $k = (1,2)$. Then $(1,1)$ is a critical point for f.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Then the functions $f(x,y)$ and $g(x,y) = (f(x,y))^2$ have the same critical points.

(c) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be smooth functions and consider a max-min problem for f with the constraint $g(x,y) = 0$. Suppose $f(x,y)$ has a maximum value at a point a relative to the constraint $g(x,y) = 0$. If the Lagrange multiplier if $\lambda = 0$, then a is also a critical point for f without the constraint.

(d) Let $A \subset \mathbb{R}^n$ be bounded and Riemann-measurable. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is bounded. Then f is integrable on A.

Answer:

		Proof or counterexample
(a)	T or F	**True**. Since f is differentiable at $a = (1,1)$, we have $f'(a) = [\alpha, \beta]$, $D_h f(1,1) = \alpha + \beta = 0$, $D_k f(1,1) = \alpha + 2\beta = 0$. It follows that $\alpha = \beta = 0$, namely $f'(a) = 0$. This means, by definition, that a is a critical point.
(b)	T or F	**False**. For example consider $f(x,y) = xy$ and $g(x,y) = x^2y^2$. It is easy to check that: - the only critical point of f is $(0,0)$, - the critical points of g are all points in the x-axis and the y-axis.
(c)	T or F	**True**. The max-min points of f relative to the constraint $g = 0$ are solutions of the equations: $g(x,y) = 0$, $\nabla f(x,y) = \lambda \nabla g(x,y)$. Hence, if $\lambda = 0$, it follows that $\nabla f(x,y) = 0$, namely (x,y) is a critical point for f even without the constraint.
(d)	T or F	**False**. For example consider $f : [0,1] \to \mathbb{R}$ be given by $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational. Then f is bounded in the interval $[0,1]$ but it is not Riemann-integrable on $[0,1]$.

Problem 3

Let $f : \mathbb{R}^4 \to \mathbb{R}$ be given by

$$f(x, y, z) \;=\; e^{xy} + xyz - x \sin z \;,$$

and let $a = (2, 0, \frac{\pi}{2})$.

(a) Find the derivative of f at a.
(b) Find the directional derivative $D_h f(a)$ in the direction of $h = (-3, 1, 2)$.
(c) Find the tangent plane Π to the graph of f at a.

Solution:

We have

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y, z) &= ye^{xy} + yz - \sin z \;, \\ \frac{\partial f}{\partial y}(x, y, z) &= xe^{xy} + xz \;, \\ \frac{\partial f}{\partial z}(x, y, z) &= xy - x \cos z \;, \end{aligned}$$

Hence

$$f'(a) \;=\; \left[\frac{\partial f}{\partial x}(a)\,,\; \frac{\partial f}{\partial y}(a)\,,\; \frac{\partial f}{\partial z}(a)\right] \;=\; [-1\,,\, 2+\pi\,,\, 0]$$

and

$$D_h f(a) \;=\; f' f(a) h \;=\; [-1\,,\, 2+\pi\,,\, 0] \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \;=\; 3 + (2+\pi) \;=\; 5 + \pi \;.$$

The graph of f is defined as the image of the function $F(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x \\ y \\ z \\ f(x, y, z) \end{bmatrix}$.

Hence the tangent plane at a, i.e. $\Pi = \left\{F(a) + F'(a)h \,|\, h \in \mathbb{R}^3\right\}$, is

$$\begin{aligned} \Pi &= \left\{ \begin{bmatrix} 2 \\ 0 \\ \pi/2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2+\pi & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} 2+x \\ y \\ \pi/2 + z \\ -1 - x + 2y + \pi y \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \end{aligned}$$

Answer:

(a)	$f'(a) =$	$(-1, 2+\pi, 0)$
(b)	$D_h f(a) =$	$5 + \pi$
(c)	$\Pi =$	$\left\{(2+x, y, \frac{\pi}{2} + z, -1 - x + 2y + \pi y) \,\middle|\, x, y, z \in \mathbb{R}\} \subset \mathbb{R}^4 \right\}$

Problem 4

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Recall that the Laplacian of f is defined as

$$\nabla^2 f \; = \; \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \; ,$$

and that f is said to be harmonic if $\nabla^2 f(x) = 0$ for every $x \in \mathbb{R}^2$. We now say that f is *subharmonic* if

$$\nabla^2 f(x) \; > \; 0 \; , \quad \forall x \in \mathbb{R}^2 \; .$$

Let D be the unit disk centered at the origin,

$$D \; = \; \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 \leq 1 \right\} \; .$$

Prove that if f is subharmonic in D, then f does not have a maximum in the interior of D.

Argument in short:

Suppose, by contradiction, $a \in D^\circ$ is a point of maximum for f.

Since f is smooth, then a is a critical point and the Hessian of f at $x = a$ is

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(a) & \frac{\partial^2 f}{\partial x \partial y}(a) \\ \frac{\partial^2 f}{\partial x \partial y}(a) & \frac{\partial^2 f}{\partial y^2}(a) \end{bmatrix} .$$

By assumption the trace $\text{Tr} H = \nabla^2 f(a) > 0$, namely the sum of the eigenvalues of H is positive, hence at least one of the eigenvalues, say λ, is positive.

But then there is a direction, i.e. the direction of the eigenvector v of λ, in which f as a local minimum.

Problem 5

Let $R \subset \mathbb{R}^3$ be a closed rectangle, and let $f : R \to \mathbb{R}$ be a continuous non-negative function, i.e. $f(x, y, z) \geq 0, \forall (x, y, z) \in R$. Prove or disprove: if there is some $a \in R^\circ$ such that $f(a) > 0$, then $\int_R f > 0$.

Answer in short:

	Proof or counterexample
T or F	**True.** Suppose $f(a) = \alpha > 0$. Since f is continuous, $\exists \epsilon > 0$ s.t. $f(x) > \alpha/2$ for every x in a cube Q centered at a of side length ϵ. Moreover, if ϵ is small enough, $Q \subset R$ (since $a \in R^\circ$). But then $Q \times [0, \frac{\alpha}{2}] \subset O_f = \{(x, y, z, f(x, y, z)) \mid (x, y, z) \in R\}$, which implies $\int_R f = v(O_f) \geq v(Q \times [0, \frac{\alpha}{2}]) = \frac{1}{2}\epsilon^3\alpha > 0$. (Note: in fact, the assumption that a is in the interior of R is not needed. It just simplifies a little bit the argument.)

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