

Math 23b: Theoretical Linear Algebra
and Multivariable Calculus I

Practice questions for the final exam

May 12, 2006

Problem 1

Decide whether the following statements are True or False. (Note: There is no need to justify your answers, just circle T or F. You get +3 for every correct answer and -1 for every wrong answer.)

T or F: If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a continuous function and $S \subset \mathbb{R}^n$ is an open set, then $f(S)$ is an open subset of \mathbb{R}^m .

T or F: The union of any collection of closed sets is closed.

T or F: If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable, then it is continuously differentiable.

Answers F, F, F

Problem 2

- (a) Provide an example of a subset $S \subset \mathbb{R}^n$ that is neither open nor closed.
- (b) Provide an example of an open cover of the interval $(0, 1)$ that has no finite subcover.
- (c) Let X be a metric space and $A \subset X$ be a subset that is both open and closed. Prove or disprove: A must be a "trivial" subset (i.e. either the empty set or the whole X).
- (d) Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = \frac{x}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Prove or disprove: f is continuous.

Answers

- (a) $(0, 1] \subset \mathbb{R}$.
- (b) $\{(\frac{1}{n}, 1) \mid n \in \mathbb{N}\}$.
- (c) False: $X = [0, 1] \cup [2, 3]$, $A = [0, 1] \subset X$.
- (d) False: $f(\frac{1}{n}, 0) = n$, which does not tend to zero.

Problem 3

Recall that if S is a subset of the metric space X , we denote by \bar{S} , S° , S^c respectively the closure, the interior and the complement of S . Let then $X = \mathbb{R}^2$ with the usual euclidean metric, and consider the following subsets:

$$A = \left\{ (x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Q} \right\}, \quad B = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \right\}.$$

Find the following sets (no justification is necessary if the answer is correct):

- (a) A° ,
- (b) $(A^c)^\circ$,
- (c) \bar{A} ,
- (d) \bar{A}^c ,
- (e) $\bar{A} \cap \bar{A}^c$,

- (f) B° ,
- (g) $(B^c)^\circ$,
- (h) \overline{B} ,
- (i) $\overline{B^c}$,
- (j) $\overline{B \cap B^c}$.
- (k) $A \cap B$.

Answers

- (a) $A^\circ = \emptyset$,
- (b) $(A^c)^\circ = \emptyset$,
- (c) $\overline{A} = \mathbb{R}^2$,
- (d) $\overline{A^c} = \mathbb{R}^2$,
- (e) $\overline{A \cap A^c} = \mathbb{R}^2$,
- (f) $B^\circ = B$,
- (g) $(B^c)^\circ = \{(x, y) \mid x^2 + y^2 > 1\}$,
- (h) $\overline{B} = \{(x, y) \mid x^2 + y^2 \leq 1\}$,
- (i) $\overline{B^c} = \{(x, y) \mid x^2 + y^2 \geq 1\}$,
- (j) $\overline{B \cap B^c} = \{(x, y) \mid x^2 + y^2 = 1\}$.
- (k) $A \cap B = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

Problem 4

Let $S = \{(x, \sin x) \mid x > 0\} \subset \mathbb{R}^2$. Find its closure \overline{S} (no justification is necessary if the answer is correct).

Answer $\overline{S} = S \cup \{(0, 0)\}$

Problem 5

Let X be a metric space, let $a \in X$ and let (x_1, x_2, \dots) be a sequence of elements of X such that $d(x_n, a) > n$ for every $n \in \mathbb{N}$. Prove that there is no limit point for the sequence.

Proof Assume by contradiction that b is a limit point for the sequence. Then there are infinitely many n 's such that $d(x_n, b) < 1$. In particular we can find $n > d(a, b) + 1$ such that $d(x_n, b) < 1$. But then by triangular inequality $n < d(a, x_n) \leq d(a, b) + d(x_n, b) < d(a, b) + 1$, which is a contradiction.

Problem 6

Suppose X is a metric space with a distance function d . Show that X is also a metric space with the new distance function \tilde{d} given by

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Proof \tilde{d} is clearly symmetric and positive definite. We only need to check that it satisfies triangular inequality:

$$\tilde{d}(a, c) \leq \tilde{d}(a, b) + \tilde{d}(b, c)$$

which is equivalent to

$$\frac{d(a, c)}{1 + d(a, c)} \leq \frac{d(a, b)}{1 + d(a, b)} + \frac{d(b, c)}{1 + d(b, c)}$$

which is equivalent to

$$d(a, c)(1 + d(a, b))(1 + d(b, c)) \leq d(a, b)(1 + d(b, c))(1 + d(a, c)) + d(b, c)(1 + d(a, b))(1 + d(a, c))$$

which is equivalent to

$$\begin{aligned} & d(a, c) + d(a, c)d(a, b) + d(a, c)d(b, c) + d(a, c)d(a, b)d(b, c) \\ & \leq d(a, b) + d(a, b)d(b, c) + d(a, b)d(a, c) \\ & \quad + d(b, c) + d(b, c)d(a, b) + d(b, c)d(a, c) + 2d(a, b)d(b, c)d(a, c) \end{aligned}$$

which is equivalent to

$$d(a, c) + \leq d(a, b) + d(a, b)d(b, c) + d(b, c) + d(b, c)d(a, b) + d(a, b)d(b, c)d(a, c)$$

which is obviously true, by triangular inequality for d .

Problem 7

Let $U, V \subset \mathbb{R}^n$ be open subsets and let $f : U \rightarrow V$ be differentiable, invertible, with inverse differentiable. As usual, let $f^{-1} : V \rightarrow U$ denote the inverse function. Prove that, if $a \in V$,

$$(f^{-1})'(a) = \left(f'(f^{-1}(a)) \right)^{-1}.$$

Proof I will give a "sloppy proof" which can be formalized by using the appropriate $\epsilon - \delta$ arguments. By definition

$$f^{-1}(a + h) \simeq f^{-1}(a) + (f^{-1})'(a)h$$

hence

$$\begin{aligned} a + h &= f(f^{-1}(a + h)) \simeq f(f^{-1}(a) + (f^{-1})'(a)h) \\ &\simeq f(f^{-1}(a)) + f'(f^{-1}(a))(f^{-1})'(a)h = a + f'(f^{-1}(a))f^{-1})'(a)h. \end{aligned}$$

We thus conclude

$$h = f'(f^{-1}(a))f^{-1})'(a)h$$

namely $f'(f^{-1}(a))f^{-1})'(a) = \mathbb{I}$, or equivalently $f^{-1})'(a) = (f'(f^{-1}(a)))^{-1}$.

Problem 8

Let $\text{Mat}_{n \times n}(\mathbb{R})$ be the vector space of $n \times n$ -matrices, and recall that we have a standard isomorphism $\text{Mat}_{n \times n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ which makes $\text{Mat}_{n \times n}(\mathbb{R})$ a Euclidean space. Consider the function $F : \text{Mat}_{n \times n}(\mathbb{R}) \rightarrow \text{Mat}_{n \times n}(\mathbb{R})$ be given by

$$F(A) = A^2.$$

Prove that the directional derivative of F in the direction of $H \in \text{Mat}_{n \times n}(\mathbb{R})$ is

$$D_H F(A) = AH + HA.$$

Proof By definition, the directional derivative is such that

$$\lim_{\|H\| \rightarrow 0} \frac{F(A + H) - F(A) - D_H F(A)}{\|H\|} = 0.$$

But we have

$$F(A + H) - F(A) - D_H F(A) = (A + H)^2 - A^2 - AH - HA = H^2,$$

so if we divide by $\|H\|$ and tend to 0 we get 0.

Problem 9

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth function such that:

- $f(1, 1, 1) = -2$,
- the vectors $A = (1, 1, 3)$ and $B = (3, 1, -1)$ are tangent to the "level surface" $S = \{(x, y, z) \mid f(x, y, z) = -2\} \subset \mathbb{R}^3$ at $(1, 1, 1)$,

- the directional derivative of f at $(1, 1, 1)$ in the direction of $(1, 0, 0)$ is 2.

Compute the directional derivative of f at $(1, 1, 1)$ in the direction of $(1, 1, 0)$.

Solution Let $v = (\alpha, \beta, \gamma) = \nabla f(1, 1, 1)$. We have by assumption $v \cdot A = v \cdot B = 0$, which implies $\alpha + \beta + 3\gamma = 0$ and $3\alpha + \beta - \gamma = 0$, namely $\beta = -\frac{5}{2}\alpha$, $\gamma = \frac{1}{2}\alpha$. From the last condition we know that $\alpha = 2$, hence $v = (2, -5, 1)$. Hence $D_{(1,1,0)}f(1, 1, 1) = (2, -5, 1) \cdot (1, 1, 0) = -3$.

Problem 10

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth function such that:

- the directional derivative at $(1, 0, 0) \in \mathbb{R}^3$ in the direction of $A = (1, 3, 1)$ is 1,
- the directional derivative at $(1, 0, 0) \in \mathbb{R}^3$ in the direction of $B = (2, 1, 3)$ is 2,
- the directional derivative at $(1, 0, 0) \in \mathbb{R}^3$ in the direction of $C = (1, 2, 3)$ is 3.

Compute $\frac{\partial f}{\partial x}(1, 0, 0)$.

Solution Let $v = (\alpha, \beta, \gamma) = \nabla f(1, 0, 0)$. We have by assumption $v \cdot A = \alpha + 3\beta + \gamma = 1$, $v \cdot B = 2\alpha + \beta + 3\gamma = 2$, $v \cdot C = \alpha + 2\beta + 3\gamma = 3$. We then get $\alpha = -\frac{7}{9}$, $\beta = \frac{2}{9}$, $\gamma = \frac{10}{9}$. Hence $\frac{\partial f}{\partial x}(1, 0, 0) = -\frac{7}{9}$.

Problem 11

Find all local and global maxima and minima of the function $f : D \rightarrow \mathbb{R}$, where D is the disk

$$D = \left\{ (x, y) \mid x^2 + y^2 \leq 4 \right\}$$

and f is given by

$$f(x, y) = \frac{x^3}{3} + \frac{y^3}{3} - \frac{x^2}{2} - \frac{y^2}{2} + 1.$$

More precisely:

- Classify all critical points in the interior of the disk D .
- Find the extrema of f on the boundary of D using the method of Lagrange multipliers.
- Determine the points of global maximum and minimum, and the corresponding values of f , on all of D .

Solution

- Critical points are solutions of $\nabla f = 0$, namely

$$x^2 - x = 0, \quad y^2 - y = 0.$$

So there are four critical points in D : $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. To find the type of critical points we use the second derivative test. The Hessian of f is

$$H = \begin{bmatrix} 2x - 1 & 0 \\ 0 & 2y - 1 \end{bmatrix}.$$

Hence $H(0, 0)$ is negative definite and $(0, 0)$ is a local maximum, $H(1, 0)$ and $H(0, 1)$ have one negative and one positive eigenvalue, hence $(1, 0)$ and $(0, 1)$ are saddle points, $H(1, 1)$ is positive definite and $(1, 1)$ is a local minimum. The corresponding max-min values of f are $f(0, 0) = 1$, $f(1, 1) = 2/3$.

- (b) By the method of Lagrange multipliers we need to solve this system of equations:

$$\nabla f = \lambda \nabla g, \quad g = 0,$$

where $g = x^2 + y^2 - 4$ gives the constraint of the max-min problem. Namely

$$x^2 - x = 2\lambda x, \quad y^2 - y = 2\lambda y, \quad x^2 + y^2 = 4.$$

From the first equation we get that either $x = 0$ or $x = 1 + 2\lambda$, similarly from the second equation we get either $y = 0$ or $y = 1 + 2\lambda$. By considering all possibilities we find again four critical points: $(0, 2)$, $(0, -2)$, $(2, 0)$, $(-2, 0)$. The corresponding values of f are: $f(2, 0) = f(0, 2) = 5/3$, $f(-2, 0) = f(0, -2) = -11/3$.

- (c) In conclusion, the points of global minimum are $(-2, 0)$, $(0, -2)$ and $f = -11/3$, while the points of global maximum are $(2, 0)$, $(0, 2)$ and $f = 5/3$.

Problem 12

Let $R \subset \mathbb{R}^n$ be a rectangle and $f : R \rightarrow \mathbb{R}$ be an integrable function. Prove that $|f|$ is also integrable and

$$\left| \int_R f \right| \leq \int_R |f|.$$

Proof By assumption f is integrable, namely f_+ and f_- are both integrable and

$$\int_R f = \int_R f_+ - \int_R f_-.$$

By definition of f_+ and f_- we have $|f| = f_+ + f_-$, hence $|f|$ is also integrable and

$$\int_R |f| = \int_R f_+ + \int_R f_-,$$

which of course implies $|\int_R f| \leq \int_R |f|$.

Problem 13

Let $R = [a, b] \times [c, d] \subset \mathbb{R}^2$ and let $f : R \rightarrow \mathbb{R}$ be a twice continuously differentiable function. Prove that

$$\int_R \frac{\partial^2 f}{\partial x \partial y} = f(b, d) + f(a, c) - f(a, d) - f(b, c).$$

Proof By Fubini's Theorem and the Fundamental Theorem of Calculus applied twice we have

$$\begin{aligned} \int_R \frac{\partial^2 f}{\partial x \partial y} &= \int_{[a, b]} \left(\int_{[c, d]} \frac{\partial^2 f}{\partial x \partial y} dy \right) dx = \int_{[a, b]} \left(\frac{\partial f}{\partial x}(x, d) - \frac{\partial f}{\partial x}(x, c) \right) dx \\ &= f(b, d) + f(a, c) - f(a, d) - f(b, c). \end{aligned}$$

Problem 14

Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded with bounded support. Suppose f^2 is integrable. Prove or disprove: f is integrable.

False Counterexample: $f : [0, 1] \rightarrow \mathbb{R}$ given by $f(x) = 1$ if x is rational and $f(x) = -1$ if x is irrational. Then f is not integrable but $f^2(x) = 1$ is obviously integrable.

Problem 15

Evaluate $\int_{\gamma} (2xyzdx + x^2zdy + x^2ydz)$, where γ is a smooth curve in \mathbb{R}^3 with initial point $(1, 1, 1)$ and final point $(1, 2, 4)$.

Solution We notice that $2xyzdx + x^2zdy + x^2ydz = df$ where $f(x, y, z) = x^2yz$. Hence by Stokes Theorem we have

$$\int_{\gamma} (2xyzdx + x^2zdy + x^2ydz) = f(1, 2, 4) - f(1, 1, 1) = 8 - 1 = 7.$$

Problem 16

Let $A = \left\{ (x, y) \in \mathbb{R}^2 \mid x, y \geq 0, \frac{1}{2} \leq x^2 + y^2 \leq 1 \right\}$, and let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by

$$f(x, y) = \frac{1}{(x^2 + y^2)^2}.$$

- sketch the region A .
- state the change of variable formula for multivariable integrals.
- compute $\int_A f$ by making an appropriate change of variables.

Solution

- Sorry, but if I am bad in drawing pictures at the blackboard, I am much worse at drawing them with a computer... so, I leave this part to your imagination!
- If $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is of class C^1 , invertible, with inverse of class C^1 , and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is integrable, then

$$\int_A f(x)dx = \int_{F^{-1}(A)} f(F(u))|\det(F'(u))|du.$$

- In our case we pick F to be polar coordinates: $F(r, \theta) = (r \cos \theta, r \sin \theta)$, so that $\det F'(r, \theta) = r$, and the domain of integration becomes $F^{-1}(A) = \{(r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2}, \frac{1}{2} \leq r \leq 1\}$. Hence

$$\begin{aligned} \int_A f(x, y)dx dy &= \int_{F^{-1}(A)} f(F(r, \theta))|\det(F'(r, \theta))|dr d\theta \\ &= \int_{[0, \pi/2]} \left(\int_{[1/2, 1]} \frac{1}{r^4} r dr \right) d\theta = \frac{\pi}{2} \int_{[1/2, 1]} r^{-3} dr = \frac{3\pi}{4} \end{aligned}$$

Problem 17

Let $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ be following smooth curve:

$$\gamma(t) = (\sin(t)e^t, \cos(t)e^t).$$

Sketch the curve γ and compute its length.

Solution

$$L(\gamma) = \int_{[0, 1]} \|\gamma'(t)\| dt = \int_{[0, 1]} \sqrt{e^{2t} \sin^2 t + e^{2t} \cos^2 t} dt = \int_{[0, 1]} e^t dt = e - 1$$

Problem 18

Let D be the region in \mathbb{R}^2 described, in polar coordinates, by the conditions

$$0 \leq \theta \leq \pi/2, 0 \leq r \leq \cos \theta.$$

- sketch the region D ,
- compute the area of D .

Solution

- (a) See part (a) of Problem 16.
 (b) As in Problem 16, we use the change of variable formula for $F(r, \theta) = (r \cos \theta, r \sin \theta)$. We get

$$\begin{aligned} A(D) &= \int_D dx dy = \int_{F^{-1}(D)} r dr d\theta \\ &= \int_{[0, \pi/2]} \left(\int_{[0, \cos \theta]} r dr \right) d\theta = \frac{1}{2} \int_{[0, \pi/2]} \cos^2 \theta d\theta = \frac{\pi}{8}. \end{aligned}$$

Problem 19

Consider the following differential 2-form in \mathbb{R}^3 :

$$\omega = xy dx dy + 2y dx dz + x dy dz .$$

Either find a differential 1-form w such that $dw = \omega$, or prove that no such a 1-form exist.

Solution By definition derivative of a differential 2-form we get

$$d\omega = \frac{\partial xy}{\partial z} dz dx dy + \frac{\partial 2y}{\partial y} dy dx dz + \frac{\partial x}{\partial x} dx dy dz = -dx dy dz \neq 0 .$$

In particular ω is not closed, and hence it cannot be exact. So w cannot exist.