Math 23b: Theoretical Linear Algebra and Multivariable Calculus I

Practice questions for the final exam

May 12, 2006

Problem 1

Decide whether the following statements are True or False. (Note: There is no need to justify your answers, just circle T or F. You get +3 for every correct answer and -1 for every wrong answer.)

T or F: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function and $S \subset \mathbb{R}^n$ is an open set, then $f(S)$ is an open subset of $\mathbb{R}^m$.

T or F: The union of any collection of closed sets is closed.

T or F: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, then it is continuously differentiable.

Answers F, F, F

Problem 2

(a) Provide an example of a subset $S \subset \mathbb{R}^n$ that is neither open nor closed.
(b) Provide an example of an open cover of the interval $(0, 1)$ that has no finite subcover.
(c) Let X be a metric space and $A \subset X$ be a subset that is both open and closed. Prove or disprove: A must be a "trivial" subset (i.e. either the empty set or the whole X).
(d) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \frac{x}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Prove or disprove: f is continuous.

Answers

(a) $(0, 1] \subset \mathbb{R}$.
(b) $\{(\frac{1}{n}, 1) \mid n \in \mathbb{N}\}$.
(c) False: $X = [0, 1] \cup [2, 3]$, $A = [0, 1] \subset X$.
(d) False: $f(\frac{1}{n}, 0) = n$, which does not tend to zero.

Problem 3

Recall that if S is a subset of the metric space X, we denote by $\bar{S}$, S°, S^c respectively the closure, the interior and the complement of S. Let then $X = \mathbb{R}^2$ with the usual euclidean metric, and consider the following subsets:

$$A = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x, y \in \mathbb{Q}\right\}, \; B = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 < 1\right\}.$$

Find the following sets (no justification is necessary if the answer is correct):

(a) A°,
(b) $(A^c)^\circ$,
(c) $\overline{A}$,
(d) $\overline{A^c}$,
(e) $\overline{A} \cap \overline{A^c}$,

(f) B°,
(g) $(B^c)^\circ$,
(h) $\overline{B}$,
(i) $\overline{B^c}$,
(j) $\overline{B} \cap \overline{B^c}$.
(k) $\overline{A \cap B}$.

Answers

(a) $A^\circ = \emptyset$,
(b) $(A^c)^\circ = \emptyset$,
(c) $\overline{A} = \mathbb{R}^2$,
(d) $\overline{A^c} = \mathbb{R}^2$,
(e) $\overline{A} \cap \overline{A^c} = \mathbb{R}^2$,
(f) $B^\circ = B$,
(g) $(B^c)^\circ = \{(x,y) \,|\, x^2 + y^2 > 1\}$,
(h) $\overline{B} = \{(x,y) \,|\, x^2 + y^2 \leq 1\}$,
(i) $\overline{B^c} = \{(x,y) \,|\, x^2 + y^2 \geq 1\}$,
(j) $\overline{B} \cap \overline{B^c} = \{(x,y) \,|\, x^2 + y^2 = 1\}$.
(k) $\overline{A \cap B} = \{(x,y) \,|\, x^2 + y^2 \leq 1\}$.

Problem 4

Let $S = \{(x, \sin x) \,|\, x > 0\} \subset \mathbb{R}^2$. Find its closure $\bar{S}$ (no justification is necessary if the answer is correct).

Answer $\overline{S} = S \cup \{(0,0)\}$

Problem 5

Let X be a metric space, let $a \in X$ and let $(x_1, x_2, \dots)$ be a sequence of elements of X such that $d(x_n, a) > n$ for every $n \in \mathbb{N}$. Prove that there is no limit point for the sequence.

Proof Assume by contradiction that b is a limit point for the sequence. Then there are infinitely many n's such that $d(x_n, b) < 1$. In particular we can find $n > d(a,b) + 1$ such that $d(x_n, b) < 1$. But then b y triangular inequality $n < d(a, x_n) \leq d(a,b) + d(x_n, b) < d(a,b) + 1$, which is a contradiction.

Problem 6

Suppose X is a metric space with a distance function d. Show that X is also a metric space with the new distance function $\tilde{d}$ given by

$$\tilde{d}(x,y) \;=\; \frac{d(x,y)}{1 + d(x,y)}$$

Proof $\tilde{d}$ is clearly symmetric and positive definite. We only need to check that it satisfies triangular inequality:

$$\tilde{d}(a,c) \leq \tilde{d}(a,b) + \tilde{d}(b,c)$$

which is equivalent to

$$\frac{d(a,c)}{1 + d(a,c)} \leq \frac{d(a,b)}{1 + d(a,b)} + \frac{d(b,c)}{1 + d(b,c)}$$

which is equivalent to

$$d(a,c)(1+d(a,b))(1+d(b,c)) \leq d(a,b)(1+d(b,c))(1+d(a,c)) + d(b,c)(1+d(a,b))(1+d(a,c))$$

which is equivalent to

$$\begin{aligned}&d(a,c)+d(a,c)d(a,b)+d(a,c)d(b,c)+d(a,c)d(a,b)d(b,c)\\&\leq d(a,b)+d(a,b)d(b,c)+d(a,b)d(a,c)\\&+d(b,c)+d(b,c)d(a,b)+d(b,c)d(a,c)+2d(a,b)d(b,c)d(a,c)\end{aligned}$$

which is equivalent to

$$d(a,c)+\leq d(a,b)+d(a,b)d(b,c)+d(b,c)+d(b,c)d(a,b)+d(a,b)d(b,c)d(a,c)$$

which is obviously true, by triangular inequality for d.

Problem 7

Let $U, V \subset \mathbb{R}^n$ be open subsets and let $f : U \to V$ be differentiable, invertible, with inverse differentiable. As usual, let $f^{-1} : V \to U$ denote the inverse function. Prove that, if $a \in V$,

$$(f^{-1})'(a) \;=\; \Big(f'\big(f^{-1}(a)\big)\Big)^{-1} \;.$$

Proof I will give a "sloppy proof" which can be formalized by using the appropriate $\epsilon - \delta$ arguments. By definition

$$f^{-1}(a+h) \simeq f^{-1}(a) + (f^{-1})'(a)h$$

hence

$$\begin{aligned}&a+h = f(f^{-1}(a+h)) \simeq f(f^{-1}(a)+(f^{-1})'(a)h)\\&\simeq f(f^{-1}(a)) + f'(f^{-1}(a))(f^{-1})'(a)h) = a + f'(f^{-1}(a))f^{-1})'(a)h\,.\end{aligned}$$

We thus conclude

$$h = f'(f^{-1}(a))f^{-1})'(a)h$$

namely $f'(f^{-1}(a))f^{-1})'(a) = \mathbb{I}$, or equivalently $f^{-1})'(a) = (f'(f^{-1}(a)))^{-1}$.

Problem 8

Let $\mathrm{Mat}_{n\times n}(\mathbb{R})$ be the vector space of $n \times n$-matrices, and recall that we have a standard isomorphism $\mathrm{Mat}_{n\times n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ which makes $\mathrm{Mat}_{n\times n}(\mathbb{R})$ a Euclidean space. Consider the function $F : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ be given by

$$F(A) \;=\; A^2 \;.$$

Prove that the directional derivative of F in the direction of $H \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is

$$D_H F(A) \;=\; AH + HA \;.$$

Proof By definition, the directional derivative is such that

$$\lim_{||H||\to 0} \frac{F(A+H) - F(A) - D_H F(A)}{||H||} = 0\,.$$

But we have

$$F(A+H) - F(A) - D_H F(A) = (A+H)^2 - A^2 - AH - HA = H^2\,,$$

so if we divide by $||H||$ and tend to 0 we get 0.

Problem 9

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function such that:

- $f(1,1,1) = -2$,
- the vectors $A = (1,1,3)$ and $B = (3,1,-1)$ are tangent to the "level surface" $S = \{(x,y,z)\,|\,f(x,y,z) = -2\} \subset \mathbb{R}^3$ at $(1,1,1)$,

- the directional derivative of f at $(1,1,1)$ in the direction of $(1,0,0)$ is 2.

Compute the directional derivative of f at $(1,1,1)$ in the direction of $(1,1,0)$.

Solution Let $v=(\alpha,\beta,\gamma)=\nabla f(1,1,1)$. We have by assumption $v\cdot A=v\cdot B=0$, which implies $\alpha+\beta+3\gamma=0$ and $3\alpha+\beta-\gamma=0$, namely $\beta=-\frac{5}{2}\alpha$, $\gamma=\frac{1}{2}\alpha$. From the last condition we know that $\alpha=2$, hence $v=(2,-5,1)$. Hence $D_{(1,1,0)}f(1,1,1)=(2,-5,1)\cdot(1,1,0)=-3$.

Problem 10

Let $f:\mathbb{R}^3\to\mathbb{R}$ be a smooth function such that:

- the directional derivative at $(1,0,0)\in\mathbb{R}^3$ in the direction of $A=(1,3,1)$ is 1,
- the directional derivative at $(1,0,0)\in\mathbb{R}^3$ in the direction of $B=(2,1,3)$ is 2,
- the directional derivative at $(1,0,0)\in\mathbb{R}^3$ in the direction of $C=(1,2,3)$ is 3.

Compute $\frac{\partial f}{\partial x}(1,0,0)$.

Solution Let $v=(\alpha,\beta,\gamma)=\nabla f(1,0,0)$. We have by assumption $v\cdot A=\alpha+3\beta+\gamma=1$, $v\cdot B=2\alpha+\beta+3\gamma=2$, $v\cdot C=\alpha+2\beta+3\gamma=3$. We then get $\alpha=-\frac{7}{9}$, $\beta=\frac{2}{9}$, $\gamma=109$. Hence $\frac{\partial f}{\partial x}(1,0,0)=-\frac{7}{9}$.

Problem 11

Find all local and global maxima and minima of the function $f:D\to\mathbb{R}$, where D is the disk

$$D=\left\{(x,y)\,\middle|\,x^2+y^2\le 4\right\}$$

and f is given by

$$f(x,y)=\frac{x^3}{3}+\frac{y^3}{3}-\frac{x^2}{2}-\frac{y^2}{2}+1\ .$$

More precisely:

(a) Classify all critical points in the interior of the disk D.
(b) Find the extrema of f on the boundary of D using the method of Lagrange multipliers.
(c) Determine the points of global maximum and minimum, and the corresponding values of f, on all of D.

Problem 12

Let $R\subset\mathbb{R}^n$ be a rectangle and $f:R\to\mathbb{R}$ be an integrable function. Prove that $|f|$ is also integrable and

$$\left|\int_R f\right|\le\int_R|f|\ .$$

Problem 13

Let $R=[a,b]\times[c,d]\subset\mathbb{R}^2$ and let $f:R\to\mathbb{R}$ be a twice continuously differentiable function. Prove that

$$\int_R\frac{\partial^2 f}{\partial x\partial y}=f(b,d)+f(a,c)-f(a,d)-f(b,c)\ .$$

Problem 14

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is bounded with bounded support. Suppose f^2 is integrable. Prove or disprove: f is integrable.

Problem 15

Evaluate $\int_\gamma (2xyzdx + x^2zdy + x^2ydz)$, where γ is a smooth curve in $\mathbb{R}^3$ with initial point $(1, 1, 1)$ and final point $(1, 2, 4)$.

Problem 16

Let $A = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x, y \geq 0,\ \frac{1}{2} \leq x^2 + y^2 \leq 1\right\}$, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) \ = \ \frac{1}{(x^2 + y^2)^2} \ .$$

(a) sketch the region A.
(b) state the change of variable formula for multivariable integrals.
(c) compute $\int_A f$ by making an appropriate change of variables.

Problem 17

Let $\gamma : [0, 1] \to \mathbb{R}^2$ be following smooth curve:

$$\gamma(t) \ = \ (\sin(t)e^t \ , \ \cos(t)e^t) \ .$$

Sketch the curve γ and compute its length.

Problem 18

Let D be the region in $\mathbb{R}^2$ described, in polar coordinates, by the conditions

$$0 \leq \theta \leq \pi/2 \, , \ 0 \leq r \leq \cos\theta \ .$$

(a) sketch the region D,
(b) compute the area of D.

Problem 19

Consider the following differential 2-form in $\mathbb{R}^3$:

$$\omega \ = \ xydxdy + 2ydxdz + xdydz \ .$$

Either find a differential 1-form w such that $dw = \omega$, or prove that no such a 1-form exist.