

Math 23b: Theoretical Linear Algebra
and Multivariable Calculus I

Practice questions for the final exam

May 12, 2006

Problem 1

Decide whether the following statements are True or False. (Note: There is no need to justify your answers, just circle T or F. You get +3 for every correct answer and -1 for every wrong answer.)

T or F: If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a continuous function and $S \subset \mathbb{R}^n$ is an open set, then $f(S)$ is an open subset of \mathbb{R}^m .

T or F: The union of any collection of closed sets is closed.

T or F: If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable, then it is continuously differentiable.

Problem 2

- (a) Provide an example of a subset $S \subset \mathbb{R}^n$ that is neither open nor closed.
- (b) Provide an example of an open cover of the interval $(0, 1)$ that has no finite subcover.
- (c) Let X be a metric space and $A \subset X$ be a subset that is both open and closed. Prove or disprove: A must be a "trivial" subset (i.e. either the empty set or the whole X).
- (d) Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = \frac{x}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Prove or disprove: f is continuous.

Problem 3

Recall that if S is a subset of the metric space X , we denote by \bar{S} , S° , S^c respectively the closure, the interior and the complement of S . Let then $X = \mathbb{R}^2$ with the usual euclidean metric, and consider the following subsets:

$$A = \left\{ (x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Q} \right\}, \quad B = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \right\}.$$

Find the following sets (no justification is necessary if the answer is correct):

- (a) A° ,
- (b) $(A^c)^\circ$,
- (c) \bar{A} ,
- (d) $\overline{A^c}$,
- (e) $\bar{A} \cap \overline{A^c}$,
- (f) B° ,
- (g) $(B^c)^\circ$,
- (h) \bar{B} ,
- (i) $\overline{B^c}$,
- (j) $\bar{B} \cap \overline{B^c}$.
- (k) $A \cap B$.

Problem 4

Let $S = \{(x, \sin x) \mid x > 0\} \subset \mathbb{R}^2$. Find its closure \bar{S} (no justification is necessary if the answer is correct).

Problem 5

Let X be a metric space, let $a \in X$ and let (x_1, x_2, \dots) be a sequence of elements of X such that $d(x_n, a) > n$ for every $n \in \mathbb{N}$. Prove that there is no limit point for the sequence.

Problem 6

Suppose X is a metric space with a distance function d . Show that X is also a metric space with the new distance function \tilde{d} given by

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Problem 7

Let $U, V \subset \mathbb{R}^n$ be open subsets and let $f : U \rightarrow V$ be differentiable, invertible, with inverse differentiable. As usual, let $f^{-1} : V \rightarrow U$ denote the inverse function. Prove that, if $a \in V$,

$$(f^{-1})'(a) = \left(f'(f^{-1}(a)) \right)^{-1} .$$

Problem 8

Let $\text{Mat}_{n \times n}(\mathbb{R})$ be the vector space of $n \times n$ -matrices, and recall that we have a standard isomorphism $\text{Mat}_{n \times n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ which makes $\text{Mat}_{n \times n}(\mathbb{R})$ a Euclidean space. Consider the function $F : \text{Mat}_{n \times n}(\mathbb{R}) \rightarrow \text{Mat}_{n \times n}(\mathbb{R})$ be given by

$$F(A) = A^2 .$$

Prove that the directional derivative of F in the direction of $H \in \text{Mat}_{n \times n}(\mathbb{R})$ is

$$D_H F(A) = AH + HA .$$

Problem 9

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth function such that:

- $f(1, 1, 1) = -2$,
- the vectors $A = (1, 1, 3)$ and $B = (3, 1, -1)$ are tangent to the "level surface" $S = \{(x, y, z) \mid f(x, y, z) = -2\} \subset \mathbb{R}^3$ at $(1, 1, 1)$,
- the directional derivative of f at $(1, 1, 1)$ in the direction of $(1, 0, 0)$ is 2.

Compute the directional derivative of f at $(1, 1, 1)$ in the direction of $(1, 1, 0)$.

Problem 10

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth function such that:

- the directional derivative at $(1, 0, 0) \in \mathbb{R}^3$ in the direction of $A = (1, 3, 1)$ is 1,
- the directional derivative at $(1, 0, 0) \in \mathbb{R}^3$ in the direction of $B = (2, 1, 3)$ is 2,
- the directional derivative at $(1, 0, 0) \in \mathbb{R}^3$ in the direction of $C = (1, 2, 3)$ is 3.

Compute $\frac{\partial f}{\partial x}(1, 0, 0)$.

Problem 11

Find all local and global maxima and minima of the function $f : D \rightarrow \mathbb{R}$, where D is the disk

$$D = \left\{ (x, y) \mid x^2 + y^2 \leq 4 \right\}$$

and f is given by

$$f(x, y) = \frac{x^3}{3} + \frac{y^3}{3} - \frac{x^2}{2} - \frac{y^2}{2} + 1.$$

More precisely:

- Classify all critical points in the interior of the disk D .
- Find the extrema of f on the boundary of D using the method of Lagrange multipliers.
- Determine the points of global maximum and minimum, and the corresponding values of f , on all of D .

Problem 12

Let $R \subset \mathbb{R}^n$ be a rectangle and $f : R \rightarrow \mathbb{R}$ be an integrable function. Prove that $|f|$ is also integrable and

$$\left| \int_R f \right| \leq \int_R |f|.$$

Problem 13

Let $R = [a, b] \times [c, d] \subset \mathbb{R}^2$ and let $f : R \rightarrow \mathbb{R}$ be a twice continuously differentiable function. Prove that

$$\int_R \frac{\partial^2 f}{\partial x \partial y} = f(b, d) + f(a, c) - f(a, d) - f(b, c).$$

Problem 14

Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded with bounded support. Suppose f^2 is integrable. Prove or disprove: f is integrable.

Problem 15

Evaluate $\int_{\gamma} (2xyz dx + x^2 z dy + x^2 y dz)$, where γ is a smooth curve in \mathbb{R}^3 with initial point $(1, 1, 1)$ and final point $(1, 2, 4)$.

Problem 16

Let $A = \left\{ (x, y) \in \mathbb{R}^2 \mid x, y \geq 0, \frac{1}{2} \leq x^2 + y^2 \leq 1 \right\}$, and let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by

$$f(x, y) = \frac{1}{(x^2 + y^2)^2}.$$

- sketch the region A .
- state the change of variable formula for multivariable integrals.
- compute $\int_A f$ by making an appropriate change of variables.

Problem 17

Let $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ be following smooth curve:

$$\gamma(t) = (\sin(t)e^t, \cos(t)e^t) .$$

Sketch the curve γ and compute its length.

Problem 18

Let D be the region in \mathbb{R}^2 described, in polar coordinates, by the conditions

$$0 \leq \theta \leq \pi/2, \quad 0 \leq r \leq \cos \theta .$$

- (a) sketch the region D ,
- (b) compute the area of D .

Problem 19

Consider the following differential 2-form in \mathbb{R}^3 :

$$\omega = xydx dy + 2ydx dz + xdy dz .$$

Either find a differential 1-form w such that $dw = \omega$, or prove that no such a 1-form exist.