Math 23b - Practice Midterm Exam 1

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Problem 1: Prove or disprove each of the following statements:

(a) Every set in $\mathbb{R}^n$ is either open or closed (or both).
False. The set $(0,1] \subset \mathbb{R}$ is neither open nor closed.

(b) Every subset of a compact set in $\mathbb{R}^n$ is compact.
False. $(0,1)$ is a subset of the compact set $[0,1]$ in $\mathbb{R}$ but it's not compact.

(c) For any subset A of a metric space X, we have $\bar{A} = \bar{\bar{A}}$.
True. It was proved in class.

(d) If $K \subset \mathbb{R}^n$ is compact, then $f(K)$ is compact.
False. f should be continuous for this to happen. For example, take $f : [0,1]$ given by $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 17$. The set $[0,1]$ is compact, but $f([0,1]) = [1, +\infty)$ is not compact.

(e) Let $A \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Then $f^{-1}(f(A)) = A$.
False. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 17 \, \forall x \in \mathbb{R}$, and let $A = [2,3] \subset \mathbb{R}$. Then $f^{-1}(f(A)) = f^{-1}(f([2,3])) = f^{-1}(\{17\}) = \mathbb{R} \neq A$.

(f) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function and let $(x_1, x_2, \dots)$ be a sequence in $\mathbb{R}^n$ converging to a. Then $(f(x_1), f(x_2), \dots)$ converges to $f(a)$.
True. This is one of the definitions of continuous functions.

(g) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function and let $A \subset \mathbb{R}^n$ be open. Then $f(A)$ is open.
False. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 17 \, \forall x \in \mathbb{R}$, and let $A = (2,3) \subset \mathbb{R}$. Then $f(A) = \{17\}$, which is not open.

Problem 2: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined as follows: $f(0,0) = 0$ and

$$f(x,y) = \frac{x^2}{x^2 + y^2} , \quad \text{for } (x,y) \neq (0,0) .$$

Indicate whether each of the following statement is true or false:

(a) f is continuous at $(0,0)$,

(b) f is differentiable at $(0,0)$.

Solution.

f is not continuous (and hence not differentiable) at $(0,0)$. Indeed we have $f(\delta, 0) = 1$ and $f(0, \delta) = 0$, so in the ball of radius $\delta > 0$ we'll always have points in which $f = 1$ and points in which $f = 0$.

Problem 3: Let X be a metric space and let A be a subset of X. Recall that, by definition, A is said to be *dense* if $\bar{A} = X$. Moreover we give the following

Definition 1. A is said to be *thin* if it is closed and it has no interior points.

(a) Give an example of a thin subset of $\mathbb{R}^2$.

(b) Prove that if A is thin, then A^c is open and dense.

Solution.

(a) $[0,1] \subset \mathbb{R}^2$ is a thin subset.

(b) Suppose $A \subset X$ is thin, i.e. it is closed and it has no interior points. Since A is closed, we know that A^c is open (we proved it in class). We are left to prove that A is dense, namely $\overline{A^c} = X$ or, equivalently, every element of X is a contact point of A^c. Clearly every element of A^c is a contact point of A^c, so we only need to prove that every point of A is a contact point of A^c. This is equivalent to say that, if $a \in A$ and $\epsilon > 0$, then $S(a,\epsilon) \cap A^c \neq \emptyset$, which is true since, by assumption, A has no interior points.

Problem 4: (a) Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Define what it means for f to be *differentiable* at $a \in \mathbb{R}^n$.

(b) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous at all $x \in \mathbb{R}^2$, but which is not differentiable at some point $a \in \mathbb{R}^2$.

(c) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is *bounded* (namely, $\exists M > 0$ such that $|f(x)| < M$, $\forall x \in \mathbb{R}^2$), and let

$$g(x,y) \;=\; xyf(x,y)\ .$$

Prove that g is differentiable at $(0,0)$. (What is the derivative $g'(0,0)$?)

Solution.

(a) f is differentiable at $a \in \mathbb{R}^n$ if there is an $m \times n$-matrix A such that

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - Ah}{||h||} \;=\; 0\ .$$

(b) For example the function $f(x,y) = |x|$ is continuous, but it is not differentiable at $(0,y)$.

(c) I claim that $g'(0,0) = [0,0]$. Indeed

$$\left| g(x,y) - g(0,0) - [0,0] \begin{bmatrix} x \\ y \end{bmatrix} \right| = |xyf(y,y)| \le |x||y|M\ .$$

We can then choose $\delta < \frac{\epsilon}{M}$. Hence, for $\left|\left| \begin{bmatrix} x \\ y \end{bmatrix} \right|\right| < \delta$, we have

$$\left| \frac{g(x,y) - g(0,0) - [0,0] \begin{bmatrix} x \\ y \end{bmatrix}}{||h||} \right| \le \frac{|x||y|M}{\sqrt{x^2+y^2}} \le \delta M < \epsilon\ .$$

This proves that

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y) - g(0,0) - [0,0] \begin{bmatrix} x \\ y \end{bmatrix}}{||h||} = 0\ ,$$

as we wanted.

Problem 5: (a) Define what it means for a set A (in a metric space X) to be compact.

(b) Suppose $f : X \to \mathbb{R}$ is continuous and let $A \subset X$ be a compact set. Show that f attains its maximum value on A.

Solution.

(a) A is compact if for every infinite set $S \subset A$ we can find $a \in A$ which is a limit point for S.

(b) This was proved in class: $f(A) \subset \mathbb{R}$ is a compact set, hence closed and bounded, hence $\sup(f(A)) \in f(A)$.