Math 23b - Practice Midterm Exam 1

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Problem 1: Prove or disprove each of the following statements:

(a) Every set in $\mathbb{R}^n$ is either open or closed (or both).

(b) Every subset of a compact set in $\mathbb{R}^n$ is compact.

(c) For any subset A of a metric space X, we have $\bar{A} = \bar{\bar{A}}$.

(d) If $K \subset \mathbb{R}^n$ is compact, then $f(K)$ is compact.

(e) Let $A \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuou function. Then $f^{-1}(f(A)) = A$.

(f) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuou function and let $(x_1, x_2, \dots)$ be a sequence in $\mathbb{R}^n$ converging to a. Then $(f(x_1), f(x_2), \dots)$ converges to $f(a)$.

(g) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuou function and let $A \subset \mathbb{R}^n$ be open. Then $f(A)$ is open.

Problem 2: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined as follows: $f(0,0) = 0$ and

$$f(x,y) = \frac{x^2}{x^2+y^2} \ , \quad \text{for } (x,y) \neq (0,0) \ .$$

Indicate whether each of the following statement is true or false:

(a) f is continuous at $(0,0)$,

(b) f is differentiable at $(0,0)$.

Problem 3: Let X be a metric space and let A be a subset of X. Recall that, by definition, A is said to be *dense* if $\bar{A} = X$. Moreover we give the following

Definition 1. A is said to be *thin* if it is closed and it has no interior points.

(a) Give an example of a thin subset of $\mathbb{R}^2$.

(b) Prove that if A is thin, then A^c is open and dense.

Problem 4: (a) Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Define what it means for f to be *differentiable* at $a \in \mathbb{R}^n$.

(b) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous at all $x \in \mathbb{R}^n$, but which is not differentiable at some point $a \in \mathbb{R}^n$.

(c) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is *bounded* (namely, $\exists M > 0$ such that $|f(x)| < M \ , \forall x \in \mathbb{R}^2$), and let

$$g(x,y) = xyf(x,y) \ .$$

Prove that g is differentiable at $(0,0)$. (What is the derivative $g'(0,0)$?)

Problem 5: (a) Define what it means for a set A (in a metric space X) to be compact.

(b) Suppose $f : X \to \mathbb{R}$ is continuous and let $A \subset X$ be a compact set. Show that f attains its maximum value on A.