Math 23bTheoretical Linear Algebra and Multivariable Calculus II

PROBLEM SET 1

Problem 1: In this problem we give a mathematical foundation to the *decimal expansion* of real numbers. Let x be a positive real number. We define an infinite sequence of integers $(n_0, n_1, n_2, \dots)$ as follows: We let n_0 be the largest non negative integer such that $n_0 \leq x$. Then we let $n_1 \in \{0, 1, \dots, 9\}$ be such that

$$n_0 + \frac{n_1}{10} \leq x < n_0 + \frac{n_1 + 1}{10} .$$

Assuming by induction we defined the integers $n_0, n_1, \dots, n_k$, we define n_{k+1} as the unique integer in $\{0, 1, \dots, 9\}$ such that

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_{k+1}}{10^{k+1}} \leq x < n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1} + 1}{10^{k+1}} .$$

(1) Convince yourself that all the integers $n_0, n_1, n_2, \dots$ are uniquely defined.

(2) Consider the sequence of real (in fact rational) numbers $(x_0, x_1, x_2, \dots)$, where

$$x_k = n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} .$$

Let $S = \{x_0, x_1, x_2, \dots\} \subset \mathbb{R}$. Prove that S is non-empty and bounded above.

(3) Prove that $x = \sup S$. (**Note:** This fact justifies the standard notation $x = n_0.n_1 n_2 n_3 \dots$ of the real number x in decimal expansion.)

Problem 2: Prove the following properties of the real numbers.

(1) $\forall x \in \mathbb{R}$, $\exists m, n \in \mathbb{Z}$ such that $m < x < n$.

(2) $\forall x \in \mathbb{R}$, $\exists! n \in \mathbb{Z}$ such that $n \leq x < n + 1$. (This number is usually called the *integral part* of x and it is denoted $[x]$)

(3) $\forall x \in \mathbb{R}$, $\exists! n \in \mathbb{Z}$ such that $x \leq n < x + 1$.

(4) $\forall x > 0$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

(5) $\forall x, y \in \mathbb{R}$ such that $x < y$, there are infinitely many distinct rational numbers $r \in \mathbb{Q}$ such that $x < r < y$.

(6) $\forall x, y \in \mathbb{R}$ such that $x < y$, there is one (in fact infinitely many) irrational number $z \in \mathbb{R} - \mathbb{Q}$ such that $x < z < y$. (Namely, irrational numbers, as the rationals, are *dense* in $\mathbb{R}$).

(7) If $s, t \in \mathbb{Q}$ and $t \neq 0$, then $s + t, st, s - t, s/t \in \mathbb{Q}$.

(8) If $s, t \in \mathbb{R} \backslash \mathbb{Q}$, what can you say about $s + t, st, s - t, s/t$?

(9) If $s \in \mathbb{Q}$, $t \in \mathbb{R} \backslash \mathbb{Q}$, what can you say about $s + t, st, s - t, s/t$?

Problem 3: Given a positive integer $n \in \mathbb{N}$, we say n is *even* if $n = 2k$ for some $k \in \mathbb{N}$, and n is *odd* if $n + 1$ is even. Prove that every $n \in \mathbb{N}$ is either even or odd, and no integer can be both even and odd.

Problem 4: (1) Let $A \subset \mathbb{R}$ be non-empty and upper bounded. Let $B = \{-a \mid a \in A\}$ Prove that B is non-empty and bounded below, and that

$$\inf B = -\sup A .$$

(2) Let A, $B \subset \mathbb{R}$ be non-empty and bounded above, and let $C = \{a + b \,|\, a \in A,\, b \in B\}$. Prove that C is non-empty and bounded above, and that

$$\sup C \;=\; \sup A + \sup B\ .$$