Math 23bTheoretical Linear Algebra and Multivariable Calculus II

PROBLEM SET 10

Problem 1: Suppose $f,\, g : \mathbb{R}^n \to \mathbb{R}$ are admissible functions, and let $c \in \mathbb{R}$. Prove that $f + g$ and cf are admissible. Namely the set of admissible functions is a vector space.

Problem 2: Let $f : A \to \mathbb{R}$ be an integrable function on $A \subset \mathbb{R}^n$. Show that $|f|$ is also integrable and

$$\left| \int_A f \right| \;\leq\; \int_A |f| \;.$$

Problem 3: Let $A = [0,1] \times [0,1]$ and define $f : A \to \mathbb{R}$ as follows:

$$f(x,y) \;=\; \begin{cases} 0 \quad , & \text{if either } x \text{ or } y \text{ is not in } \mathbb{Q} \\ \frac{1}{q} \quad , & \text{if } x, y \in \mathbb{Q} \text{ and } y = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

Show that f is integrable. (What is $\int_A f$?)

Problem 4: Let $f,\, g : A \to \mathbb{R}$ be two bounded functions such that the set

$$C \;=\; \{x \in A \,|\, f(x) \neq g(x)\} \;,$$

is measurable with volume zero. Prove that f is integrable on A if and only if g is, and

$$\int_A f \;=\; \int_A g \;.$$

Problem 5: Let $f : R^n \to \mathbb{R}$ be a function continuous at $x \in \mathbb{R}^n$. Let $Q(x,\epsilon)$ denote the cube of center x and sides of length 2ϵ, i.e.

$$Q_\epsilon = Q(x,\epsilon) \;=\; [x_1 - \epsilon, x_1 + \epsilon] \times \cdots \times [x_n - \epsilon, x_n + \epsilon] \;.$$

Prove that

$$\lim_{\epsilon \to 0} \frac{1}{v(Q_\epsilon)} \int_{Q_\epsilon} f \;=\; f(x) \;.$$

(Note: this is a multivariable analogue of the Fundamental Theorem of calculus. Why?)

Problem 6: Let $f : [a,b] \to \mathbb{R}$ be a bounded increasing function, i.e. $f(y) > f(x)$ if $y > x$. Prove that f is integrable on $[a,b]$.