Math 23bTheoretical Linear Algebra and Multivariable Calculus II

PROBLEM SET 11

Problem 1: Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two function bounded, with bounded support and integrable.

(a) Prove that f^2 is integrable.

(b) Prove that fg is integrable.

(*Hint:* (a) We may assume f is non-negative (why?). Given $\epsilon > 0$, let h and k be step functions such that $h \leq f \leq k$ and $\int k - \int h < \frac{\epsilon}{M}$, where M is an upper bound for $|h(x) + k(x)|$. Then prove that h^2 and k^2 are step functions such that $h^2 \leq f^2 \leq k^2$, and $\int k^2 - \int h^2 < \epsilon$. Deduce that f^2 is integrable. (b) Express fg in terms of $(f+g)^2$ and $(f-g)^2$ and use part (a).)

Problem 2: (a) Consider the following function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \cap [0,1] \text{ and } x = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{otherwise} \end{cases} .$$

Prove that f is bounded, with bounded support and integrable, but it is not admissible.

(b) Consider the following function $F : \mathbb{R}^2 \to \mathbb{R}$,

$$F(x,y) = \begin{cases} \frac{1}{q} & \text{if } x, y \in \mathbb{Q} \cap [0,1] \text{ and } y = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{otherwise} \end{cases} .$$

Prove that:

1. $\int_{\mathbb{R}^2} F(x,y)dxdy = 0$,
2. $\int_{\mathbb{R}} F(x,y)dy = 0$ for all $x \in \mathbb{R}$,
3. $\int_{\mathbb{R}} F(x,y)dx = 0$ for all rational y but it is not defined for y rational.

Does this contradict Fubini's Theorem?

Problem 3: Use the transformation defined by

$$u = \frac{2x}{x^2+y^2}\ ,\ v = \frac{2y}{x^2+y^2}\ ,$$

to evaluate

$$\int_R \frac{dxdy}{(x^2+y^2)^2}\ ,$$

where R is the region in the first quadrant of the xy-plane which is bounded by the circles $x^2+y^2 = 6x$, $x^2+y^2 = 4x$, $x^2+y^2 = 8y$, $x^2+y^2 = 2y$. (Draw a picture of the region R).

Problem 4: Compute the value of the following iterated integrals by transforming them in polar coordinates

(a) $\int_0^1 \left(\int_{x^2}^{x} \frac{1}{\sqrt{x^2+y^2}} dy \right) dx$,

(b) $\int_0^a \left(\int_0^{\sqrt{a^2-y^2}} (x^2+y^2) dx \right) dy$.

Problem 5: In this exercise we want to prove the famous formula

$$\int_{-\infty}^{\infty} e^{-x^2} dx \;=\; \sqrt{\pi}\ . \tag{1}$$

(The integrating function is called "Gauss function", and it is very important in Probability Theory).

(a) Let $I(r) = \int_{-r}^{r} e^{-u^2} du$. Show that $I(r)^2 = \int\int_R e^{-x^2-y^2} dxdy$, where $R = [-r,r] \times [-r,r]$.

(b) If $B(r)$ is a circular disk of radius r, show that

$$\int\int_{B(r)} e^{-x^2-y^2} dxdy \;\leq\; I(r)^2 \;\leq\; \int\int_{B(2r)} e^{-x^2-y^2} dxdy\ .$$

(c) Compute $\int\int_{B(r)} e^{-x^2-y^2} dxdy$ using polar coordinates.

(d) Deduce that $\lim_{r\to\infty} I(r) = \sqrt{\pi}$, namely (1) holds.

Problem 6: In this problem we want to compute the volume of n–dimensional unit ball, $B_n = B_n(1)$, where $B_n(r) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, 0 \leq ||x|| \leq r,\ i = 1, \dots, n\} \subset \mathbb{R}^n$. In particular, we want to prove that

$$v(B_{2n}) = \frac{\pi^n}{n!} \quad , \quad v(B_{2n+1}) = \frac{2^{n+1}\pi^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}\ . \tag{2}$$

(a) Use a change of variable $y_i = x_i/r$ to prove that $v(B_n(r)) = r^n v(B_n)$.

(b) Prove that $v(B_1) = 2$, $v(B_2) = \pi$ (for B_2 you can use polar coordinates).

(c) Prove the identity

$$\chi_{B_n}(x_1, \dots, x_n) = \chi_{B_2}(x_1, x_2)\chi_{B_{n-2}\left(\sqrt{1-x_1^2-x_2^2}\right)}(x_3, \dots, x_n)\ .$$

(d) Use polar coordinates to prove that

$$\int_{B_2} (1 - x_1^2 - x_2^2)^{(n-2)/2} dx_1 dx_2 = \frac{2\pi}{n}\ .$$

(e) Use Fubini's Theorem and parts (c) and (d) to show that

$$v(B_n) = \frac{2\pi}{n} v(B_{n-2})\ .$$

(f) Use (e) to prove equations (2) by induction.