Math 23bTheoretical Linear Algebra and Multivariable Calculus II

PROBLEM SET 12

Problem 1: Compute $\int_\gamma 2xyzdx + x^2zdy + x^2ydz$, where γ is a smooth curve in $\mathbb{R}^3$ starting at $(1,1,1)$ end ending at $(1,2,4)$.

Problem 2: Use the definition of surface integral to prove the formula of the area of a sphere $S(r)$ of radius r in $\mathbb{R}^3$: $A(S(r)) = 4\pi r^2$.

Problem 3: Prove that the alternating multilinear k-forms $dx_{i_1} \dots dx_{i_k}$ with $1 \le i_1 < \cdots < i_k \le n$ are linearly independent (Hence they form a basis of the space of alternating multilinear k-forms $\wedge^k \mathbb{R}^{1\times n}$, which we'll call the "standard basis").

Problem 4: A smooth *differential k-form* on $\mathbb{R}^n$ is, by definition, a smooth function $\omega : \mathbb{R}^n \to \wedge^k \mathbb{R}^{1\times n}$. Namely, in terms of the standard basis of $\wedge^k \mathbb{R}^{1\times n}$, we have

$$\omega(x_1, \dots, x_n) = \sum_{1\le i_1<\cdots<i_k\le n} a_{(i_1\dots i_k)}(x_1,\dots,x_n) dx_{i_1}\dots dx_{i_k}$$

for smooth real valued n-variable functions $a_{(i_1\dots i_k)}(x_1,\dots,x_n)$, $1 \le i_1 < \cdots < i_k \le n$. We will denote here by $\Omega^k(\mathbb{R}^n)$ the space of all smooth differential k-forms on $\mathbb{R}^n$.

We then define the *differential* $d : \Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$. Namely the differential of a k-form ω is the following $k+1$-form on $\mathbb{R}^n$:

$$d\omega(x_1, \dots, x_n) = \sum_{1\le i_1<\cdots<i_k\le n} \sum_{j=1}^{n} \frac{\partial a_{(i_1\dots i_k)}(x_1,\dots,x_n)}{\partial x_j} dx_j dx_{i_1}\dots dx_{i_k}\ .$$

Consider now smooth differential k-forms in $\mathbb{R}^3$.

(a) Find an explicit isomorphism between $\Omega^0(\mathbb{R}^3)$ and the space $C^\infty(\mathbb{R}^3, \mathbb{R})$ of smooth functions $f(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$.

(b) Use the standard basis of $\wedge^1 \mathbb{R}^{1\times 3}$ to find an explicit isomorphism between $\Omega^1(\mathbb{R}^3)$ and the space $C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ of smooth functions $F(x,y,z) = [f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)] : \mathbb{R}^3 \to \mathbb{R}^3$.

(c) Use the standard basis of $\wedge^2 \mathbb{R}^{1\times 3}$ to find an explicit isomorphism between $\Omega^2(\mathbb{R}^3)$ and $C^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

(d) Use the standard basis of $\wedge^3 \mathbb{R}^{1\times 3}$ to find an explicit isomorphism between $\Omega^3(\mathbb{R}^3)$ and $C^\infty(\mathbb{R}^3, \mathbb{R})$.

(e) Describe the differential $d : \Omega^0(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ explicitly, using the above isomorphisms in (a) and (b), as a function $d : C^\infty(\mathbb{R}^3, \mathbb{R}) \to C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ (Note: you should get the *gradient* of functions).

(f) Describe the differential $d : \Omega^1(\mathbb{R}^n) \to \Omega^2(\mathbb{R}^n)$ explicitly, using the above isomorphisms in (b) and (c), as a function $d : C^\infty(\mathbb{R}^3, \mathbb{R}^3) \to C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ (Note: this is usually called the *curl* of a vector field).

(g) Describe the differential $d : \Omega^2(\mathbb{R}^n) \to \Omega^3(\mathbb{R}^n)$ explicitly, using the above isomorphism in (c) and (d), as a function $d : C^\infty(\mathbb{R}^3, \mathbb{R}) \to C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ (Note: this is usually called the *divergence* of a vector field).