Math 23bTheoretical Linear Algebra and Multivariable Calculus II

PROBLEM SET 2

Problem 1: Prove that the *equinumerosity* between sets, $A \overset{\sim}{\sim} B$, is an equivalence "concept" (we cannot talk of equivalence relation, since it's not defined in a set, but in the family of all sets, which we know is not a set). Namely it is reflexive, symmetric and transitive.

Problem 2: Prove that the following sets are *countable*:

(1) An arbitrary countable union of countable sets,

(2) The set of all real "algebraic" numbers. By definition, a real number $x \in \mathbb{R}$ is said to be algebraic if it is a solution of a polynomial equation with integer coefficients, i.e. if there are $k \in \mathbb{N}$, $n_0, n_1, \dots, n_k \in \mathbb{Z}$ such that

$$n_0 x^k + n_1 x^{k-1} + \cdots + n_{k-1} x + n_k \ = \ 0 \ .$$

(**Hint:** You might find it useful to show first that the set $\{n_0, n_1, \dots, n_k \in \mathbb{Z} \mid k + |n_0| + |n_1| + \cdots + |n_k| \le N\}$ is finite.)

Problem 3: (1) Prove that the set $\{0,1\}^{\mathbb{N}} = \{(x_1, x_2, \dots) \mid x_k = 0 \text{ or } 1\}$, of all infinite sequences of 0's and 1's, is uncountable.

(2) Prove that $\{0,1\}^{\mathbb{N}} \overset{\sim}{\sim} \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

(**Hints:** for (1), recall how we proved that $\mathbb{R}$ is uncountable; for (2), recall how we proved that $\mathbb{N} \overset{\sim}{\sim} 2\mathbb{N}$.)

(**Comment:** Since, as I explained wednesday, $\{0,1\}^{\mathbb{N}} \overset{\sim}{\sim} \mathbb{R}$, this shows in particular that $\mathbb{R} \overset{\sim}{\sim} \mathbb{R}^2$, and therefore that the interval $(0,1)$ is equinumerous to the open disk of radius 1 in $\mathbb{R}^2$, which was a question which came up on wednesday.)

Problem 4: Consider the set $C[a,b]$ of all continuous functions $f : [a,b] \to \mathbb{R}$, together with the distance

$$d(f,g) \ = \ \sup\left\{ |f(x) - g(x)| \ \middle| \ a \le x \le b \right\} \ .$$

Prove that it is a metric space.

Problem 5: Consider the space $\mathbb{R}^2$ with discrete metric. Let $M \subset \mathbb{R}^2$ and $a \in \mathbb{R}^2$.

(1) When is a a contact point of M?
(2) When is a a limit point of M?
(3) When is a an interior point of M?
(4) When is a an isolated point of M?
(5) When is M closed?
(6) When is M open?

Find necessary and sufficient conditions. Justify your answers.

Problem 6: Prove that, in any metric space X,

(1) an open ball $S(a,r)$ is an open set,
(2) a closed ball $S[a,r]$ is a closed set.
(3) if $M_1 \subset M_2$, then $\overline{M_1} \subset \overline{M_2}$,

Here $\overline{M}$ denotes the closure of the set M.