Math 23bTheoretical Linear Algebra and Multivariable Calculus II

PROBLEM SET 3

Problem 1: Let $(x_1, x_2, \dots)$ be a sequence in a metric space X. Prove or disprove the following statements:

(1) if there is a unique limit point x for the sequence, then the sequence converges to x,
(2) if x is a limit point of the set $\{x_n \mid n \in \mathbb{N}\}$, then x is a limit point for the sequence $(x_1, x_2, \dots)$,
(3) if x is a limit point for the sequence $(x_1, x_2, \dots)$, then x is a limit point of the set $\{x_n \mid n \in \mathbb{N}\}$,

Problem 2: Prove the following:

Proposition 0.1. The sequence $(x_1, x_2, \dots)$ converges to x, if and only if for every open set $U \subset X$ such that $x \in U$ we have $x_n \in U$ for all but finitely many n's.

Problem 3: Let $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$ be sequences in $\mathbb{R}$ converging to x and y respectively. Prove that

(1) the sequence $(x_1y_1, x_2y_2, \dots)$ converges to xy,
(2) if $x_1, x_2, \dots$ and x are all non-zero, then the sequence $(\frac{1}{x_1}, \frac{1}{x_2}, \dots)$ converges to $\frac{1}{x}$.

Problem 4: Let $(\mathbf{x_1} = \begin{bmatrix} x_1^1 \\ \vdots \\ x_k^1 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} x_1^2 \\ \vdots \\ x_k^2 \end{bmatrix}, \dots)$ be a sequence in $\mathbb{R}^k$.

Prove that $(\mathbf{x_1}, \mathbf{x_2}, \dots)$ converges in $\mathbb{R}^k$ if and only if each of the sequences $(x_i^1, x_i^2, \dots)$, $i = 1, \dots, k$, converges in $\mathbb{R}$ and, in this case, we have

$$\lim_{n\to\infty} \mathbf{x_n} = \begin{bmatrix} \lim_{n\to\infty} x_1^n \\ \vdots \\ \lim_{n\to\infty} x_k^n \end{bmatrix}.$$

Problem 5: Let $(x_1, x_2, \dots)$ be a sequence in the metric space X. We say that the sequence is *bounded* if there exist $a \in X$ and $R > 0$ such that $x_n \in S(a, R)$ for all $n \in \mathbb{N}$. Prove or disprove the following statements:

(1) if a sequence is converging, then it is bounded,
(2) if a sequence is bounded, then it is converging.

Problem 6: Let $\mathbb{R}^{n,e}$ be the space $\mathbb{R}^n$ with the usual Euclidean metrics, and let $R^{n,d}$ be the same space with discrete metrics.

(1) When is a function $f : \mathbb{R}^{n,e} \to \mathbb{R}^{n,d}$ continuous?
(2) When is a function $f : \mathbb{R}^{n,d} \to \mathbb{R}^{n,e}$ continuous?