Math 23bTheoretical Linear Algebra and Multivariable Calculus II

PROBLEM SET 4

Problem 1: Let X be a metric space, $a \in X$, and let f be a function $f : X \to \mathbb{R}$.

Definition 0.1. We say that the limit of f for x going to a is (plus or minus) infinity, denoted

$$\lim_{x \to a} f(x) = \infty \ ,$$

if $\forall M > 0 \, \exists \delta > 0$ such that for $d(x, a) < \delta$ we have $|f(x)| > M$.

Prove or disprove the following statements:

(1) If $\lim_{x \to a} f(x) = \infty$, then $\lim_{x \to a} \frac{1}{f(x)} = 0$.

(2) If $\lim_{x \to a} f(x) = 0$, then $\lim_{x \to a} \frac{1}{f(x)} = \infty$.

Problem 2: Let $f(x, y) = \frac{2x^2 y}{x^4 + y^2}$, unless $x = y = 0$, and let $f(0, 0) = 0$.

(1) If $\phi(t) = (t, at)$, prove that $\lim_{t \to 0} f(\phi(t)) = 0$, namely f is continuous at $(0, 0)$ on any straight line through $(0, 0)$.

(2) If $\psi(t) = (t, t^2)$, prove that $\lim_{t \to 0} f(\psi(t)) = 1$, and conclude that f is not continuous at $(0, 0)$.

Problem 3: Define $f : [0, 1] \to \mathbb{R}$ as follows:

$$f(x) \ = \ \begin{cases} 0 & , \quad \text{if } x \notin \mathbb{Q} \ , \\ \frac{1}{q} & , \quad \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

(1) after numbering the rational numbers in $[0, 1]$ in any way, (namely, after finding an explicit isomorphism $\mathbb{N} \to \mathbb{Q} \cap [0, 1]$), plot $f : \mathbb{Q} \cap [0, 1] \to \mathbb{R}$.

(2) Prove that f is not continuous at any rational x.

(3) Prove that f is continuous at every irrational x.

Problem 4: We said in class that compact is the same as closed and bounded only in $\mathbb{R}^n$. In this problem we'll see that this is not true on every metric space.

Consider the metric space

$$L_2 = \left\{ X = (x_1, x_2, \dots) \mid x_n \in \mathbb{R} \ , \ \sum_{n=1}^{\infty} x_n^2 < \infty \right\}$$

with metric given by

$$d(X, Y) = \sqrt{\sum_{n=1}^{\infty} (y_n - x_n)^2}$$

(You don't need to prove that this is indeed a metric space) Let $0 = (0, 0, \dots) \in L_2$. Prove that $S[0, 1] \subset L_2$ is closed and bounded, but it is not compact.

Problem 5: In this problem we introduce another definition of compact sets, and we will show that it implies our definition (we will not prove it is in fact equivalent, even tough it is true).

Let X be a metric space and let K be a subset of X.

Definition 0.2. A collection of subsets of X, $\mathcal{C} = \{S_\alpha \mid \alpha \in \mathcal{I}\} \subset 2^X$, is said to be an *open cover* of K if

(1) S_α is open for every $\alpha \in \mathcal{I}$,
(2) $K \subset \bigcup_{\alpha \in \mathcal{I}} S_\alpha$.

A *subcover* is a subset $\Sigma \subset \mathcal{C}$ which is still an open cover of K.

Definition 0.3. K is said to be *compact** if for every open cover of K there exists a finite subcover

(Note: we will write compact* instead of compact to distinguish it from the definition we have seen in class). In this problem we want to prove the following

Theorem 0.4. *If K is compact*, then it is compact.*

Follow the following steps:

(1) Suppose that $S \subset K$ is an infinite set with no limit points in K. Prove that for every $q \in K$ there is an open ball B_q centered at q such that $B_q \cap S =$ either $\emptyset$ or $\{q\}$.
(2) Show that $\mathcal{C} = \{B_q \mid q \in K\}$ is an open cover of K which has no finite subcover.
(3) Deduce the Theorem.